# Unified Systems of FB-SPDEs/FB-SDEs with Jumps/Skew Reflections and Stochastic Differential Games [1]

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**Abstract**

We study four systems and their interactions. First, we formulate a unified system of coupled forward-backward stochastic *partial* differential equations (FB-SPDEs) with Lévy jumps, whose drift, diffusion, and jump coefficients may involve partial differential operators. A solution to the FB-SPDEs is defined by a 4-tuple general dimensional random vector-field process evolving in time together with position parameters over a domain (e.g., a hyperbox or a manifold). Under an infinite sequence of generalized local linear growth and Lipschitz conditions, the well-posedness of an adapted 4-tuple strong solution is proved over a suitably constructed topological space. Second, we consider a unified system of FB-SDEs, a special form of the FB-SPDEs, however, with *skew* boundary reflections. Under randomized linear growth and Lipschitz conditions together with a general completely-$\mathcal{S}$ condition on reflections, we prove the well-posedness of an adapted 6-tuple weak solution with *boundary regulators* to the FB-SDEs by the Skorohod problem and an oscillation inequality. Particularly, if the spectral radii in some sense for reflection matrices are strictly less than the unity, an adapted 6-tuple strong solution is concerned. Third, we formulate a stochastic differential game (SDG) with general number of players based on the FB-SDEs. By a solution to the FB-SPDEs, we get a solution to the FB-SDEs under a given control rule and then obtain a Pareto optimal Nash equilibrium policy process to the SDG. Fourth, we study the applications of the FB-SPDEs/FB-SDEs in queueing systems and quantum statistics while we use them to motivate the SDG.

**Key words and phrases:** stochastic (partial/ordinary) differential equation, Lévy jump, skew reflection, completely-$\mathcal{S}$ condition, Skorohod problem, oscillation inequality, stochastic differential game, Pareto optimal Nash equilibrium, queueing network

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## 1 Introduction

We study four systems and their interactions: a unified system of coupled forward-backward stochastic partial differential equations (FB-SPDEs) with Lévy jumps; a unified system of FB-SDEs, a special form of the FB-SPDEs, however, with skew reflections; a stochastic differential game (SDG) problem with general number of players based on the FB-SDEs; and a system of queues and their associated reflecting diffusion approximations. More precisely, there are four interconnected and streamlined aims involved in our discussions.

The first aim is to study the adapted 4-tuple strong solution $(U, V, \bar{V}, \tilde{V})$ to the unified system of coupled FB-SPDEs with Lévy jumps with respect to time-position parameter $(t, x) \in R_+ \times D$,

$$
(1.1) \qquad \left\{\begin{array}{ll} U(t,x) & = G(x) + \int_0^t \mathcal{L}(s^-, x, U, V, \bar{V}, \tilde{V}) ds \\ & \qquad + \int_0^t \mathcal{J}(s^-, x, U, V, \bar{V}, \tilde{V}) dW(s) \\ & \qquad + \int_0^t \int_{\mathcal{Z}^h} \mathcal{I}(s^-, x, U, V, \bar{V}, \tilde{V}, z) \tilde{N}(\lambda ds, dz), \\ V(t,x) & = H(x) + \int_t^\tau \bar{\mathcal{L}}(s^-, x, U, V, \bar{V}, \tilde{V}) ds \\ & \qquad + \int_t^\tau \bar{\mathcal{J}}(s^-, x, U, V, \bar{V}, \tilde{V}) dW(s) \\ & \qquad + \int_t^\tau \int_{\mathcal{Z}^h} \bar{\mathcal{I}}(s^-, x, U, V, \bar{V}, \tilde{V}, z) \tilde{N}(\lambda ds, dz), \end{array}\right.
$$

where, $t \in [0, \tau]$ and $\tau \in [0, T]$ is a stopping time with regard to a filtration defined later in the paper, $\mathcal{Z}^h = R^h - \{0\}$ or $R^h_+$ for a positive integer $h$, and $s^-$ denotes the corresponding left limit at time point $s$. In particular, $D \in R^p$ with a given $p \in \mathcal{N} = \{1, 2, ...\}$ is a connected domain, for examples, a $p$-dimensional box, a $p$-dimensional ball (or a general manifold), a $p$-dimensional sphere (or a general Riemannian manifold), or the whole Euclidean space $R^p$ of real numbers itself. The F-SPDE in (1.1) is with the given initial random vector-field $G$, while the B-SPDE in (1.1) has the known terminal random vector-field $H$. In (1.1), $U$ and $V$ are $r$-dimensional and $q$-dimensional random vector-field processes respectively, $W$ is a standard $d$-dimensional Brownian motion, and $\tilde{N}$ is a $h$-dimensional centered Lévy jump process (or centered subordinator). Furthermore, the partial differential operators of $r$-dimensional vector $\mathcal{L}$, $r \times d$-dimensional matrix $\mathcal{J}$, and $r \times h$-dimensional matrix $\mathcal{I}$ are functionals of $U$, $V$, $\bar{V}$, $\tilde{V}$, and their partial derivatives of up to the $k$th order for $k \in \{0, 1, 2, 3, ...\}$. So do the partial differential operators of $q$-dimensional vector $\bar{\mathcal{L}}$, $q \times d$-dimensional matrix $\bar{\mathcal{J}}$, and $q \times h$-dimensional matrix $\bar{\mathcal{I}}$. More precisely, for each $\mathcal{A} \in \{\mathcal{L}, \mathcal{J}, \bar{\mathcal{L}}, \bar{\mathcal{J}}\}$,

$$
(1.2) \qquad \begin{aligned} \mathcal{A}(s, x, U, V, \bar{V}, \tilde{V}) \quad \equiv \quad & \mathcal{A}(s, x, (U, \frac{\partial U}{\partial x_1}, ..., \frac{\partial^k U}{\partial x_1^{i_1} ... \partial x_p^{i_p}})(s, x), \\ & (V, \frac{\partial V}{\partial x_1}, ..., \frac{\partial^k V}{\partial x_1^{i_1} ... \partial x_p^{i_p}})(s, x), \\ & (\bar{V}, \frac{\partial \bar{V}}{\partial x_1}, ..., \frac{\partial^k \bar{V}}{\partial x_1^{i_1} ... \partial x_p^{i_p}})(s, x), \end{aligned}
$$

$$(\tilde{V}, \frac{\partial \tilde{V}}{\partial x_1}, ..., \frac{\partial^k \tilde{V}}{\partial x_1^{i_1} ... \partial x_p^{i_p}})(s, x, \cdot), \cdot),$$

where the dot "$\cdot$" in $\tilde{V}(s, x, \cdot)$ and its associated partial derivatives denotes the integration in terms of the so-called Lévy measure. However, if $\mathcal{A} \in \{\mathcal{I}, \bar{\mathcal{I}}\}$, the last line on the right-hand side of (1.2) should be changed to the form,

$$(\tilde{V}, \frac{\partial \tilde{V}}{\partial x_1}, ..., \frac{\partial^k \tilde{V}}{\partial x_1^{i_1} ... \partial x_p^{i_p}})(s, x, z), z, \cdot).$$

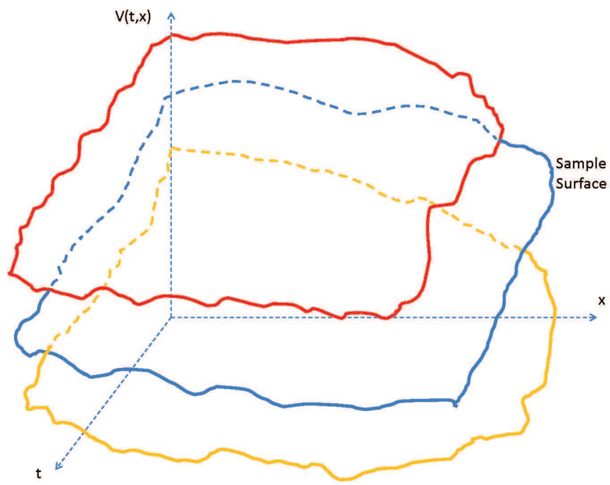


Figure 1: Sample Surface Solution to the FB-SPDEs

Note that our partial differential operators presented in (1.2) can be general-nonlinear and high-order, e.g.,

$$\mathcal{A}(s, x, U, V, \bar{V}, \tilde{V}) = f(\frac{\partial^k V(t, x)}{\partial x_1^{i_1} ... \partial x_p^{i_p}})$$

for a general nonlinear functional $f$, where $r, i_1, ..., i_p$ are nonnegative integers satisfying $i_1 + ... + i_p = k$ with $k \in \{0, 1, 2, 3, ...\}$. Furthermore, the initial random vector-field $G$, the terminal random vector-filed $H$, and the 4-tuple solution process $(U, V, \bar{V}, \tilde{V})$ can be complex-valued.

Under an infinite sequence of generalized local linear growth and Lipschitz conditions, we prove the existence and uniqueness of an adapted 4-tuple strong solution to the FB-SPDEs in a suitably constructed functional topological space. The solution to the unified system in (1.1) can be interpreted in a sample surface manner with time-position parameter $(t, x)$ (see, e.g., $V(t, x)$ in Figure 1 for such an example). The quite involved technical proof developed

in this paper is extended from our earlier work summarized in Dai [17] (arxiv, 2011) for a unified B-SPDE.

More precisely, the newly unified system of coupled FB-SPDEs in (1.1) covers many existing forward and/or backward SDEs/SPDEs as special cases, where the partial differential operators are taken to be special forms. For examples, specific single-dimensional strongly nonlinear F-SPDE and B-SPDE driven solely by Brownian motions can be respectively derived for the purpose of optimal-utility based portfolio choice (see, e.g, Musiela and Zariphopoulou [38]). Here, the strong nonlinearity is in the sense addressed by Lions and Souganidis [34] and Pardoux [42]. Furthermore, the single-dimensional stochastic Hamilton-Jacobi-Bellman (HJB) equations are also examples of our unified system in (1.1), which are specific B-SPDEs (see, e.g., ∅ksendal *et al.* [41] and references therein). Note that the proof of the well-posedness concerning solution to the B-SPDE derived in Musiela and Zariphopoulou [38] and solution to the HJB equation derived in ∅ksendal *et al.* [41] is covered by the study in Dai [17] (arxiv, 2011) although the authors in both [38] and ∅ksendal *et al.* [41] claim it as an open problem. The proof of the well-posedness about solution to the F-SPDE derived in Musiela and Zariphopoulou [38] is covered by the even more unified discussion for the coupled FB-SPDEs in (1.1) of this paper. Actually, partial motivations to enhance the unified B-SPDE in Dai [17] (arxiv, 2011) to the coupled FB-SPDEs in (1.1) are from the conference discussion [22] during 45 minutes invited talk presented by Zariphopoulou in ICM 2014, where the current author claimed that the well-posedness of solution to the F-SPDE in [38] can be proved by the method developed in Dai [17] (arxiv, 2011). Besides these existing examples, our motivations to study the coupled FB-SPDEs in (1.1) are also from optimal portfolio management in finance (see, e.g., Dai [16, 20]), and multi-channel (or multi-valued) image regularization such as color images in computer vision and network applications (see, e.g., Caselles *et al.* [7]). In this part, we also show the usages of our unified system in (1.1) in heat diffusions and quantum Hall/anomalous Hall effects as two illustrative examples to support our first aim. Mathematically, we refine a stochastic Dirichlet-Poisson problem from heat diffusions and use stochastic Schrödinger equation as model for Hall effects in quantum statistics.

It is worth to point out that the proving methodology developed in the current paper and its early version in Dai [17] (arxiv, 2011) is aimed to provide a general theory and framework to show the well-posedness of a unified general system class of the coupled FB-SPDEs in (1.1). However, some specific forms of the FB-SPDEs in (1.1) (either in forward manner or in backward manner) may be solved by alternative techniques, e.g., the author in his Fields Metal awarded work (Hairer [26] and ICM 2014) solves the KPZ equation by rough path technology, and furthermore, the related rough path theory can deal with the lack of either temporal or spatial regularity (see, e.g., Hairer [26] and reference therein).

The second aim of the paper is to prove the well-posedness of an adapted 6-tuple weak solution $((X,Y),(V,\bar{V},\tilde{V},F))$ with 2-tuple boundary regulator $(Y,F)$ to the (possible) non-Markovian system of coupled FB-SDEs with Lévy jumps and skew reflections under a given

control rule $u$,

$$
(1.3)\quad \begin{cases} \begin{cases} X(t) &= b(t^-,X(t^-),V(t^-),\bar{V}(t^-),\tilde{V}(t^-,\cdot),u(t^-,X(t^-),\cdot)dt \\ & \quad +\sigma(t^-,X(t^-),V(t^-),\bar{V}(t^-),\tilde{V}(t^-,\cdot),u(t^-,X(t^-)),\cdot)dW(t) \\ & \quad +\int_{\mathcal{Z}^h}\eta(t^-,X(t^-),V(t^-),\bar{V}(t^-),\tilde{V}(t^-,z),u(t^-,X(t^-)),z,\cdot)\tilde{N}(dt,dz) \\ & \quad +RdY(t), \\ X(0) &= x, \\ Y_i(t) &= \int_0^t I_{D_i}(X(s))dY_i(s); \end{cases} \\ \begin{cases} V(t) &= c(t^-,X(t^-),V(t^-),\bar{V}(t^-),\tilde{V}(t^-,\cdot),u(t^-,X(t^-),\cdot)dt \\ & \quad -\alpha(t^-,X(t^-),V(t^-),\bar{V}(t^-),\tilde{V}(t^-,\cdot),u(t^-,X(t^-)),\cdot)dW(t) \\ & \quad -\int_{\mathcal{Z}^h}\zeta(t^-,X(t^-),V(t^-),\bar{V}(t^-),\tilde{V}(t^-,z),u(t^-,X(t^-)),z,\cdot)\tilde{N}(dt,dz) \\ & \quad -SdF(t), \\ V(T) &= H(X(T),\cdot), \\ F_i(t) &= \int_0^t I_{\bar{D}_i}(V(s))dF_i(s). \end{cases} \end{cases}
$$

In (1.3), $X$ is a $p$-dimensional process governed by the F-SDE with skew reflection matrix $R$ and $V$ is a $q$-dimensional process governed by the B-SDE with skew reflection matrix $S$. Furthermore, $Y$ can increase only when $X$ is on a boundary $D_i$ with $i \in \{1,...,b\}$ and $F$ can increase only when $V$ is on a boundary $\bar{D}_i$ with $\in \{1,...,\bar{b}\}$, where $b$ and $\bar{b}$ are two nonnegative integers. Both $Y$ and $F$ are the regulating processes with possible jumps to push $X$ and $V$ back into the state spaces $D$ and $\bar{D}$ respectively. They are parts of the 6-tuple solution to (1.3) and determined by solution pairs to the well-known Skorohod problem (see, e.g., Dai [14], Dai and Dai [12], or Section 6 of the current paper for such a definition). Thus, we call them as Skorohod regulators (see, Figure 2 for such an example). Note that, comparing with the unified system in (1.1), the coefficients appeared in (1.3) do not contain any partial derivative operator but the FB-SDEs themselves involve skew boundary reflections. The proof for the well-posedness of an adapted 6-tuple weak solution to the FB-SDEs is based on two general conditions. The first one is a general completely-$\mathcal{S}$ condition (see, e.g., Dai [14], Dai and Dai [12], and Figure 2 for an illustration). The non-uniqueness of solution to an associated Skorohod problem under this condition is one of the major difficulties in the proof. The second one is the generalized linear growth and Lipschitz conditions, where the conventional growth and Lipschitz constant is replaced by a possible unbounded but mean-squarely integrable adapted stochastic process (see, e.g., Dai [16, 20]). In particular, if the completely-$\mathcal{S}$ condition becomes more strict, e.g., with additional requirements that the spectral radii in certain sense for both reflection matrices are strictly less than the unity, a unique adapted 6-tuple strong solution will be concerned.

Concerning coupled FB-SDEs, it motivates a hot research area (see, e.g., ∅ksendal *et al.* [41] about the discussion of coupled FB-SDEs with no boundary reflection, Karatzas and Li [31] about the study of Brownian motion driven B-SDE with reflection, and references therein). However, to our best knowledge, the coupled system in (1.3) with double skew reflection matrices and the well-posedness study in terms of an adapted 6-tuple weak solution with Lévy jumps and under a general completely-$\mathcal{S}$ condition through the Skorohod problem

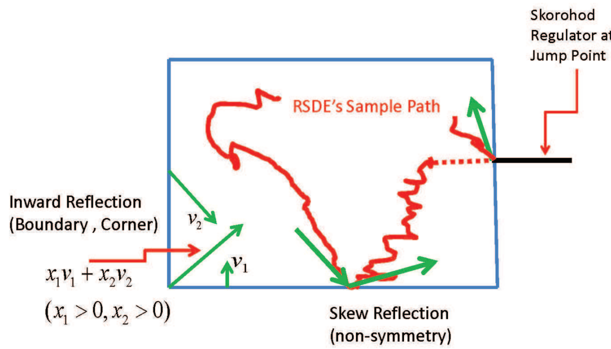


Figure 2: Skew and Inward Reflection with Skorohod Regulator under Completely-$\mathcal{S}$ Condition

are new and for the first time in this area.

The third aim of the paper involves two folds. On the one hand, we use the 4-tuple solution to the coupled FB-SPDEs in (1.1) to obtain an adapted 6-tuple solution to the system in (1.3); On the other hand, we use the obtained adapted 6-tuple solution to determine a Pareto optimal Nash equilibrium policy process to a non-zero-sum SDG problem in (1.4), which is newly formulated by the FB-SDEs in (1.3). In this game, there are $q$-players and each player $l \in \{1, ..., q\}$ has his own value function $V_l^u$ subject to the system in (1.3) under an admissible control policy $u$. Every player $l$ chooses an optimal policy to maximize his own value function over an admissible policy set $\mathcal{C}$ while the summation of all value functions is also maximized, i.e.,

$$\sup_{u \in \mathcal{C}} V_l^u(0) = V_l^{u^*}(0) \tag{1.4}$$

for each $l \in \{0, 1, ..., q\}\}$, where,

$$V_0^u(t) = \sum_{l=1}^{q} V_l^u(t). \tag{1.5}$$

Note that the total value function $V_0^u(0)$ does not have to be a constant (e.g., zero), or in other words, the game is not necessarily a zero-sum one.

The contribution and literature review of the study associated with the game in (1.4)-(1.5) for the third aim can be summarized as follows. One of the important solution methods for SDE based optimal control is the dynamic programming. In general, this method is related to a special case of the unified system in (1.1) (or its earlier unified B-SPDE form in Dai [17] (arxiv, 2011)), e.g., the specific B-SPDE with $q = 1$ (called stochastic HJB equation) in

Peng [44] with no jumps and ∅ksendal *et al.* [41] with jumps. Here, we extend the discussions in Peng [44] and ∅ksendal *et al.* [41] to a system of generalized coupled forward-backward oriented stochastic HJB equations with jumps corresponding to the case that $q > 1$. More importantly, this system provides an effective way to resolve a non-zero-sum SDG problem with jumps and general number of $q$ players, which subjects to a non-Markovian system of coupled FB-SDEs with Lévy jumps and skew reflections (see, e.g., Figure 3 for such a game platform (partially adapted from Dai [15])). By a solution to the FB-SPDEs in (1.1), we

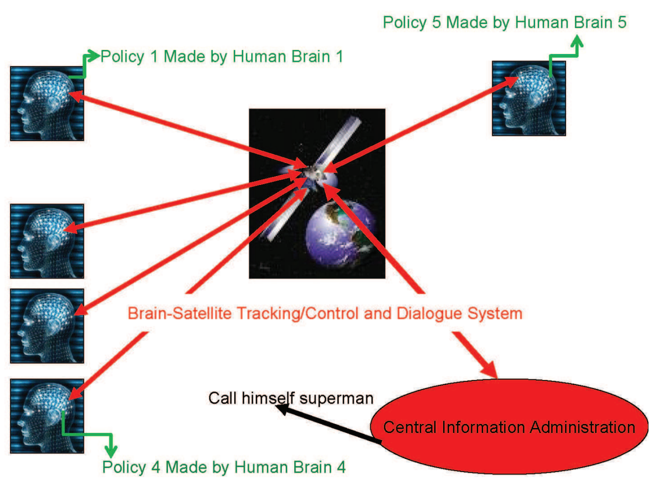


Figure 3: A 5-player game platform based on brain and satellite communication

determine a solution to the FB-SDEs in (1.3) under a given control rule and then obtain a Pareto optimal Nash equilibrium policy process to the non-zero-sum SDG problem in (1.4). Note that, the concept and technique concerning the non-zero-sum SDG and Pareto optimality used in this paper is refined and generalized from Dai [18] and Karatzas and Li [31].

The fourth aim of the paper involves three folds. First,we study some queueing networks (see, e.g., Figure 4) whose dynamics (e.g., queue length process) is governed by specific forms of the FB-SDEs in (1.3). These forms can be a Lévy driven SDE, a $p$-dimensional reflecting Brownian motion (RBM) (see, e.g., Dai [14], Dai and Dai [12], Dai and Jiang [21]), or a reflecting diffusion with regime switching (RDRS) (see, e.g., Dai [18]). The reflecting diffusion is the functional limit of a sequence of physical queueing processes under diffusive scaling, a general completely-$\mathcal{S}$ boundary reflection constraint, and a well-known heavy traffic condition (an analogous treatment as the one for "infinite constant" in the KPZ equation (see, e.g., Hairer [26])). In reality, the characteristics of Lévy driven networks may be used to model or approximate more general batch-arrival and batch-service queueing networks. Second, we discuss how to use the queueing systems and their associated reflecting diffusion approximations to motivate the SDG problem. The criterion for the players in the game

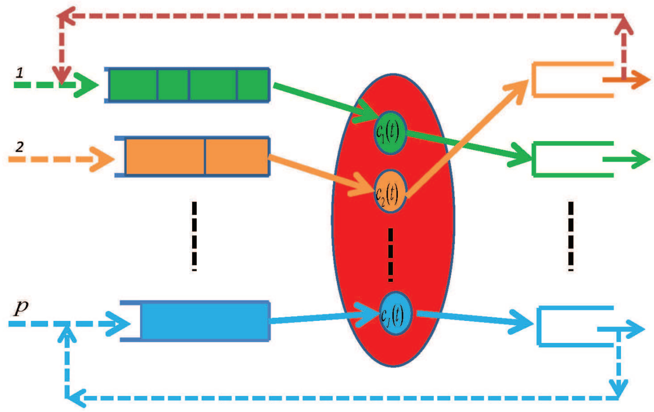


Figure 4: A queueing network system with $p$-job classes

can be the queue length based performance optimization ones or queueing related cost/profit optimization ones. Third, we study the applications of the FB-SPDEs presented by (1.1) in the queueing networks. There are two types of equations involved. One is the Kolmogorov's equation or Fokker-Planck's formula oriented PDEs/SPDEs, which are corresponding to the distributions of queueing length processes under given network control rules. This type of equations are mainly used to estimate the performance measures of the queueing networks. Another type of equations are the HJB equation oriented PDES/SPDES, which are mainly used to obtain optimal control rules over the set of admissible strategies for the queueing networks.

The remainder of the paper is organized as follows. In Section 2, we introduce suitable functional topological space and state conditions required for our main theorems to guarantee the well-posedness of an adapted 4-tuple strong solution to the unified system of coupled FB-SPDEs in (1.1). In Section 3, we study the unified system of coupled FB-SDEs with Lévy jumps and skew reflections in (1.3) and present the well-posedness theorem. In particular, we establish the solution connection between the FB-SPDEs and the FB-SDEs. In Section 4, we formally formulate the FB-SDEs based SGE problem in (1.4) and determine the Pareto optimal Nash equilibrium policy process by a system of generalized stochastic HJB equations (a particular form of the coupled FB-SPDEs). Related applications in queueing networks are also discussed. Finally, in Sections 5-7, we develop theory to prove our main theorems.

## 2 The Unified System of Coupled FB-SPDEs with Lévy Jumps

First of all, let $(\Omega, \mathcal{F}, P)$ be a fixed complete probability space. Then, we define a standard $d$-dimensional Brownian motion $W \equiv \{W(t), t \in [0, T]\}$ for a given $T \in [0, \infty)$ with

$W(t) = (W_1(t), ..., W_d(t))'$ and a $h$-dimensional general Lévy pure jump process (or special subordinator) $L \equiv \{L(t), t \in [0,T]\}$ with $L(t) \equiv (L_1(t), ..., L_h(t))'$ on the space (see, e.g., Applebaum [1], Bertoin [6], and Sato [48]). Note that the prime appeared in this paper is used to denote the corresponding transpose of a matrix or a vector. Furthermore, $W$, $L$, and their components are supposed to be independent of each other. For each $\lambda = (\lambda_1, ...\lambda_h)' > 0$, which is called a reversion rate vector in many applications, we let $L(\lambda s) = (L_1(\lambda_1 s), ..., L_h(\lambda_h s))'$. Then, we denote a filtration by $\{\mathcal{F}_t\}_{t\geq 0}$ with $\mathcal{F}_t \equiv \sigma\{\mathcal{G}, W(s), L(\lambda s) : 0 \leq s \leq t\}$ for each $t \in [0,T]$, where $\mathcal{G}$ is $\sigma$-algebra independent of $W$ and $L$. In addition, let $I_A(\cdot)$ be the index function over the set $A$ and $\nu_i$ for each $i \in \{1, ..., h\}$ be a Lévy measure. Then, we use $N_i((0,t] \times A) \equiv \sum_{0<s\leq t} I_A(L_i(s) - L_i(s^-))$ to denote a Poisson random measure with a deterministic, time-homogeneous intensity measure $ds\nu_i(dz_i)$. Thus, each $L_i$ can be represented by (see, e.g., Theorem 13.4 and Corollary 13.7 in pages 237 and 239 of Kallenberg [30])

$$L_i(t) = a_i(t) + \int_{(0,t]} \int_{\mathcal{Z}} z_i N_i(\lambda_i ds, dz_i), \ t \geq 0. \tag{2.1}$$

For convenience, we take the constant $a_i$ to be zero.

In the subsequent two subsections, we first study the unified system in (1.1), which is real-valued with a closed position parametric domain. Then, we extend the discussion to a complex-valued system with an open position parametric domain.

## 2.1 The Real-Valued System with Closed Position Parametric Domain

We let $D \in R^p$ be a closed position parametric domain and use $C^k(D, R^l)$ for each $k \in \mathcal{N}$ and $l \in \{r, q\}$ to denote the Banach space of all functions $f$ having continuous derivatives up to the order $k$ with the uniform norm for each $f$ in this space,

$$\|f\|_{C^k(D,l)} = \max_{c\in\{0,1,...,k\}} \max_{j\in\{1,...,r(c)\}} \sup_{x\in D} \left| f_j^{(c)}(x) \right|. \tag{2.2}$$

The $r(c)$ in (2.2) for each $c \in \{0, 1, ..., k\}$ is the total number of the partial derivatives of the order $c$

$$f^{(c)}_{r,i_1...i_p}(x) = \frac{\partial^c f_r(x)}{\partial x_1^{i_1} ... \partial x_p^{i_p}} \tag{2.3}$$

with $i_l \in \{0, 1, ..., c\}$, $l \in \{1, ..., p\}$, $r \in \{1, ..., l\}$, and $i_1 + ... + i_p = c$. Here, we remark that, whenever the partial derivative on the boundary $\partial D$ is concerned, it is defined in a one-side manner. In addition, let

$$f^{(c)}_{i_1,...,i_p} \equiv (f^{(c)}_{1,i_1,...,i_p}, ..., f^{(c)}_{q,i_1,...,i_p}), \tag{2.4}$$

$$f^{(c)}(x) \equiv (f^{(c)}_1(x), ..., f^{(c)}_{r(c)}(x)), \tag{2.5}$$

where each $j \in \{1, ..., r(c)\}$ corresponds to a $p$-tuple $(i_1, ..., i_p)$ and a $r \in \{1, ..., l\}$. Then, we use $C^\infty(D, R^l)$ to denote the Banach space

$$C^\infty(D, R^l) \equiv \left\{ f \in \bigcap_{c=0}^{\infty} C^c(D, R^l), \|f\|_{C^\infty(D,l)} < \infty \right\}, \tag{2.6}$$

where

$$\|f\|^2_{C^\infty(D,q)} = \sum_{c=0}^{\infty} \xi(c)\|f\|^2_{C^c(D,l)} \tag{2.7}$$

for some discrete function $\xi(c)$ in terms of $c \in \{0, 1, 2, ...\}$, which is fast decaying in $c$. For convenience, we take $\xi(c) = \frac{1}{((c^{10})!)(\eta(c)!)e^c}$ with

$$\eta(c) = [\max\{|x_1| + ... + |x_p|, x \in D\}]^c,$$

where the notation [] denotes the summation of the unity and the integer part of a real number.

Next, let $L^2_{\mathcal{F}}([0,T], C^\infty(D; R^l))$ denote the set of all $R^l$-valued (or called $C^\infty(D; R^l)$-valued) measurable random vector-field processes $Z(t,x)$ adapted to $\{\mathcal{F}_t, t \in [0,T]\}$ for each $x \in D$, which are in $C^\infty(D, R^l)$ for each fixed $t \in [0,T]$), such that

$$E\left[\int_0^T \|Z(t)\|^2_{C^\infty(D,l)} dt\right] < \infty. \tag{2.8}$$

In particular, let $L^2_{\mathcal{G}_l}(\Omega, C^\infty(D; R^l))$ with $l \in \{r, q\}$ denote the set of all $R^l$-valued random vector-fields $\zeta(x)$ that are $\mathcal{G}_l$-measurable for each $x \in D$ and satisfy

$$\|\zeta\|^2_{L^2_{\mathcal{G}}(\Omega, C^\infty(D,R^l))} \equiv E\left[\|\zeta\|^2_{C^\infty(D,l)}\right] < \infty, \tag{2.9}$$

where $\mathcal{G}_r = \mathcal{G}$ and $\mathcal{G}_q = \mathcal{F}_T$. In addition, let $L^2_p([0,T] \times \mathcal{Z}^h,\ C^\infty(D, R^{l\times h}))$ be the set of all $R^{l\times h}$-valued random vector-field processes denoted by $\tilde{V}(t,x,z) = (\tilde{V}_1(t,x,z_1), ..., \tilde{V}_h(t,x,z_h))$, which are predictable for each $x \in D$ and $z \in \mathcal{Z}^h$ and are endowed with the norm

$$E\left[\sum_{i=1}^{h} \int_0^T \int_{\mathcal{Z}} \left\|\tilde{V}_i(t,z_i)\right\|^2_{C^\infty(D,l)} \nu_i(dz_i)dt\right] < \infty. \tag{2.10}$$

Thus, we can define

$$\begin{aligned} \mathcal{Q}^2_{\mathcal{F}}([0,T] \times D) \quad \equiv \quad & L^2_{\mathcal{F}}([0,T], C^\infty(D, R^r)) \\ & \times L^2_{\mathcal{F}}([0,T], C^\infty(D, R^q)) \\ & \times L^2_{\mathcal{F},p}([0,T], C^\infty(D, R^{q\times d})) \\ & \times L^2_p([0,T] \times \mathcal{Z}^h, C^\infty(D, R^{q\times h})). \end{aligned} \tag{2.11}$$

Finally, let

$$\begin{aligned} & L^2_\nu(\mathcal{Z}^h, C^c(D, R^{q\times h})) \\ & \equiv \left\{\tilde{v} : \mathcal{Z}^h \to C^c(D, R^{q\times h}),\ \sum_{i=1}^{h} \int_{\mathcal{Z}} \|\tilde{v}_i(z_i)\|^2_{C^c(D,q)} \nu_i(dz_i) < \infty\right\} \end{aligned} \tag{2.12}$$

that is endowed with the norm

$$\|\tilde{v}\|_{\nu,c}^2 \equiv \sum_{i=1}^{h} \int_{\mathcal{Z}} \|\tilde{v}_i(z_i)\|_{C^c(D,q)}^2 \, \lambda_i \nu_i(dz_i) \tag{2.13}$$

for any $\tilde{v} \in L_\nu^2(\mathcal{Z}^h, C^c(D, R^{q\times h}))$ and $c \in \{0, 1, ..., \infty\}$. Furthermore, define

$$\begin{aligned} \mathcal{V}^\infty(D) \;\; \equiv \;\; & C^\infty(D, R^r) \\ & \times C^\infty(D, R^q) \\ & \times C^\infty(D, R^{q\times d}) \\ & \times \bar{L}_\nu^2(\mathcal{Z}^h, C^\infty(D, R^{q\times h})). \end{aligned} \tag{2.14}$$

In the sequel, we let $\|A\|$ be the largest absolute value of entries (or components) of the given matrix (or vector) $A$. Furthermore, for each $s \in [0, T]$ and $z \in \mathcal{Z}^h$, we let

$$\tilde{N}(\lambda ds, dz) \;\; = \;\; (\tilde{N}_1(\lambda_1 ds, dz_1), ..., \tilde{N}_h(\lambda_h ds, dz_h))', \tag{2.15}$$

where

$$\tilde{N}_i(\lambda_i ds, dz_i) = N_i(\lambda_i ds, dz_i) - \lambda_i ds \nu_i(dz_i) \tag{2.16}$$

for each $i \in \{1, ..., h\}$. Then, we impose some conditions to guarantee the unique existence of an adapted 4-tuple strong solution to the unified system in (1.1).

First, for each $\mathcal{A} \in \{\mathcal{L}, \bar{\mathcal{L}}, \mathcal{J}, \bar{\mathcal{J}}\}$, every $c \in \{0, 1, 2, ...,\}$, and any $(u^i, v^i, \bar{v}^i, \tilde{v}^i) \in \mathcal{V}^\infty(D)$ with $i \in \{1, 2\}$, we define

$$\begin{aligned} & \Delta\mathcal{A}^{(c)}(s, x, u^1, v^1, \bar{v}^1, \tilde{v}^1, u^2, v^2, \bar{v}^2, \tilde{v}^2) \\ & \equiv \mathcal{A}^{(c)}(s, x, u^1, v^1, \bar{v}^1, \tilde{v}^1) - \mathcal{A}^{(c)}(s, x, u^2, v^2, \bar{v}^2, \tilde{v}^2). \end{aligned}$$

Then, we assume that the generalized local Lipschitz condition is true almost surely (a.s.),

$$\begin{aligned} & \left\| \Delta\mathcal{A}^{(c+l+o)}(s, x, u^1, v^1, \bar{v}^1, \tilde{v}^1, u^2, v^2, \bar{v}^2, \tilde{v}^2) \right\| \\ & \leq K_{D,c} \Big( \|u^1 - u^2\|_{C^{k+c}(D,r)} + \|v^1 - v^2\|_{C^{k+c}(D,q)} \\ & \qquad + \|\bar{v}^1 - \bar{v}^2\|_{C^{k+c}(D,qd)} + \|\tilde{v}^1 - \tilde{v}^2\|_{\nu,k+c} \Big). \end{aligned} \tag{2.17}$$

Note that $K_{D,c}$ in (2.17) with each $c \in \{0, 1, 2, ...\}$ is a nonnegative constant. It depends on the domain $D$ and the differential order $c$ and may be unbounded as $c \to \infty$ and $D \to R^p$. $l \in \{0, 1, 2\}$ denotes the $l$th order of partial derivative of $\Delta\mathcal{A}^{(c)}(s, x, u, v, \bar{v}, \tilde{v})$ in time variable $t$. $o \in \{0, 1, 2\}$ denotes the $o$th order of partial derivative of $\Delta\mathcal{A}^{(c+l)}(s, x, u, v, \bar{v}, \tilde{v})$ in terms of a component of $u$, $v$, $\bar{v}$, or $\tilde{v}$. Furthermore, for each $\mathcal{A} \in \{\mathcal{I}, \bar{\mathcal{I}}\}$, we suppose that

$$\begin{aligned} & \sum_{i=1}^{h} \int_{\mathcal{Z}} \left\| \Delta\mathcal{A}_i^{(c+l+o)}(s, x, u^1, v^1, \bar{v}^1, \tilde{v}^1, u^2, v^2, \bar{v}^2, \tilde{v}^2, z_i) \right\|^2 \lambda_i \nu_i(dz_i) \\ & \leq K_{D,c} \Big( \|u^1 - u^2\|_{C^{k+c}(D,r)}^2 + \|v^1 - v^2\|_{C^{k+c}(D,q)}^2 \\ & \qquad + \|\bar{v}^1 - \bar{v}^2\|_{C^{k+c}(D,qd)}^2 + \|\tilde{v}^1 - \tilde{v}^2\|_{\nu,k+c}^2 \Big), \end{aligned} \tag{2.18}$$

where $\mathcal{A}_i$ is the $i$th column of $\mathcal{A}$.

Second, for each $\mathcal{A} \in \{\mathcal{L}, \bar{\mathcal{L}}, \mathcal{J}, \bar{\mathcal{J}}\}$, every $c \in \{0, 1, 2, ..., \}$, and any $(u, v, \bar{v}, \tilde{v}) \in \mathcal{V}^\infty(D)$, we suppose that the generalized local linear growth condition holds

$$\begin{aligned} &\left\|\mathcal{A}^{(c+l+o)}(s, x, u, v, \bar{v}, \tilde{v})\right\| \\ &\leq K_{D,c}\left(\delta_{0c} + \|u\|_{C^{k+c}(D,r)} + \|v\|_{C^{k+c}(D,q)}\right. \\ &\qquad \left.+\|\bar{v}\|_{C^{k+c}(D,qd)} + \|\tilde{v}\|_{\nu,k+c}\right), \end{aligned} \tag{2.19}$$

where, $\delta_{0c} = 1$ if $c = 0$ and $\delta_{0c} = 0$ if $c > 0$. Similarly, for each $\mathcal{A} \in \{\mathcal{I}, \bar{\mathcal{I}}\}$, we suppose that

$$\begin{aligned} &\sum_{i=1}^{h} \int_{\mathcal{Z}} \left\|\mathcal{A}_i^{(c+l+o)}(s, x, u, v, \bar{v}, \tilde{v}, z_i)\right\|^2 \lambda_i \nu(dz_i) \\ &\leq K_{D,c}\left(\delta_{0c} + \|u\|^2_{C^{k+c}(D,r)} + \|v\|^2_{C^{k+c}(D,q)}\right. \\ &\qquad \left.+\|\bar{v}\|^2_{C^{k+c}(D,qd)} + \|\tilde{v}\|^2_{\nu,k+c}\right). \end{aligned} \tag{2.20}$$

Then, we can state our main theorem of this subsection as follows.

**Theorem 2.1** *Suppose that $(G, H) \in L^2_{\mathcal{G}}(\Omega, C^\infty(D; R^r)) \times L^2_{\mathcal{F}_T}(\Omega, C^\infty(D; R^q))$ and conditions in (2.17)-(2.20) are true. Furthermore, assume that each $\mathcal{A} \in \{\mathcal{L}, \bar{\mathcal{L}}, \mathcal{J}, \bar{\mathcal{J}}, \mathcal{I}, \bar{\mathcal{I}}\}$ is $\{\mathcal{F}_t\}$-adapted for every fixed $x \in D$, $z \in \mathcal{Z}^h$, and any given $(u, v, \bar{v}, \tilde{v}) \in \mathcal{V}^\infty(D)$ with*

$$\mathcal{L}(\cdot, x, 0) \in L^2_{\mathcal{F}}\left([0, T], C^\infty(D, R^r)\right), \tag{2.21}$$

$$\mathcal{J}(\cdot, x, 0) \in L^2_{\mathcal{F}}\left([0, T], C^\infty(D, R^{r\times d})\right), \tag{2.22}$$

$$\mathcal{I}(\cdot, x, 0, \cdot) \in L^2_{\mathcal{F}}\left([0, T] \times R^h_+, C^\infty(D, R^{r\times h})\right), \tag{2.23}$$

$$\bar{\mathcal{L}}(\cdot, x, 0) \in L^2_{\mathcal{F}}\left([0, T], C^\infty(D, R^q)\right), \tag{2.24}$$

$$\bar{\mathcal{J}}(\cdot, x, 0) \in L^2_{\mathcal{F}}\left([0, T], C^\infty(D, R^{q\times d})\right), \tag{2.25}$$

$$\bar{\mathcal{I}}(\cdot, x, 0, \cdot) \in L^2_{\mathcal{F}}\left([0, T] \times R^h_+, C^\infty(D, R^{q\times h})\right). \tag{2.26}$$

*Then, there exists a unique adapted 4-tuple strong solution to the system in (1.1), i.e.,*

$$(U, V, \bar{V}, \tilde{V}) \in \mathcal{Q}^2_{\mathcal{F}}([0, T] \times D), \tag{2.27}$$

*and $(U, V)(\cdot, x)$ is càdlàg for each $x \in D$ almost surely (a.s.).*

The proof of Theorem 2.1 is provided in Section 5.

## 2.2 The Complex-Valued System with Open Position Parametric Domain

In this subsection, we generalize the study in Subsection 2.1 to the case corresponding to an open (or partially open) position parametric domain $D$ (e.g., $R^p$ or $R^p_+$). More exactly,

we assume that there exists a sequence of nondecreasing closed and connected sets $\{D_n, n \in \{0,1,...\}\}$ such that

$$D = \bigcup_{n=0}^{\infty} D_n. \tag{2.28}$$

Furthermore, let $C^\infty(D, \mathbb{C}^l)$ with $l \in \{r, q\}$ be the Banach space endowed with the norm for each $f$ in the space

$$\|f\|^2_{C^\infty(D,l)} \equiv \sum_{n=0}^{\infty} \xi(n+1)\|f\|^2_{C^\infty(D_n,l)}, \tag{2.29}$$

where $\mathbb{C}^l$ is the $l$-dimensional complex Euclidean space and the norm $\|f\|^2_{C^\infty(D_n,l)}$ in (2.29) is interpreted in the corresponding complex-valued sense. In addition, define $\bar{Q}^2_{\mathcal{F}}([0,\tau] \times D)$ to be the corresponding space in (2.11) if the terminal time $T$ is replaced by the stopping time $\tau$ in (1.1) and the norm in (2.7) is substituted by the one in (2.29). Finally, we use the same way to interpret the spaces $L^2_{\mathcal{G}}(\Omega, C^\infty(D; R^r))$ and $L^2_{\mathcal{F}_\tau}(\Omega, C^\infty(D; R^q))$. Then, we have the following theorem.

**Theorem 2.2** *Suppose that $(G, H) \in L^2_{\mathcal{G}}(\Omega, C^\infty(D; R^r)) \times L^2_{\mathcal{F}_\tau}(\Omega, C^\infty(D; R^q))$ and the system in (1.1) satisfies the conditions in (2.17)-(2.20) over $D_n$ for each $n \in \{0,1,...\}$ with associated (local) linear growth and Lipshitz constant $K_{D_n,c}$. Furthermore, assume that each $\mathcal{A} \in \{\mathcal{L}, \bar{\mathcal{L}}, \mathcal{J}, \bar{\mathcal{J}}, \mathcal{I}, \bar{\mathcal{I}}\}$ is $\{\mathcal{F}_t\}$-adapted for every fixed $x \in D$, $z \in \mathcal{Z}^h$, and any given $(u, v, \bar{v}, \tilde{v}) \in \mathcal{V}^\infty(D)$ with conditions in (2.21)-(2.26) being true. Then, the system in (1.1) has a unique adapted 4-tuple strong solution*

$$(U, V, \bar{V}, \tilde{V}) \in \bar{\mathcal{Q}}^2_{\mathcal{F}}([0,\tau] \times D), \tag{2.30}$$

*and $(U, V)(\cdot, x)$ is càdlàg for each $x \in D$ a.s.*

The proof of Theorem 2.2 is provided in Section 5.

### 2.3 Illustrative Examples

In this subsection, we present two illustrative examples related to heat diffusions and quantum Hall/anomalous Hall effects. Mathematically, the heat diffusions are modeled as a stochastic Dirichlet-Poisson problem and the Hall effects are presented by a stochastic Schrödinger equation. Examples related to queueing systems and SDGs will be presented in Section 4 after studying the system of coupled FB-SDEs in (1.3).

- **Heat Diffusions: Stochastic Dirichlet-Poisson Problem**

For this example, we consider a special B-SPDE (a specific backward part of the system in (1.1)). More precisely, the associated partial differential operators are given by

$$
\begin{aligned}
\bar{\mathcal{L}}(t,x,U,V,\bar{V},\tilde{V}) &\equiv -g(t,x)+\frac{1}{2}\sum_{j=1}^{p}\frac{\partial^2 V(t,x)}{\partial x_j^2},\\
\bar{\mathcal{J}}(t,x,U,V,\bar{V},\tilde{V}) &\equiv 0,\\
\bar{\mathcal{I}}(t,x,U,V,\bar{V},\tilde{V},z) &\equiv 0,
\end{aligned}
$$

where $V$ is single-dimensional (i.e., $q=1$) and $g(t,x)$ is some integrable function. Then, we can obtain the corresponding B-SPDE with jumps as follows,

$$
\begin{aligned}
V(t,x) &= H(x)+\int_t^{\tau}\left(-g(t,x)+\frac{1}{2}\sum_{j=1}^{p}\frac{\partial^2 V(t,x)}{\partial x_j^2}\right)ds \\
&\quad -\int_t^{\tau}\bar{V}(s,x)dW(s) \\
&\quad -\int_t^{\tau}\int_{z>0}\tilde{V}(s^-,x,z)\tilde{N}(\lambda ds,dz).
\end{aligned}
\tag{2.31}
$$

When the terminal value $H(x)$ is a boundary based condition over $D$, we will call the related resolution problem *a stochastic Dirichlet-Poisson problem with jumps*, which is a randomized general form of the classical Dirichlet-Poisson problem (see, e.g., the definition of a classic case in $\emptyset$ksendal [39]). An explanation about how to use this problem to estimate the inner or surface temperature of certain material or an object (e.g., Sun) is displayed in Figure 5. Physically, the randomized heat equation in (2.31) is derived from a particle system just like its classic counterpart, which can be modeled by a diffusion process as follows,

$$
dX_t = b(X_t)dt+\sigma(X_t)dW(t). \tag{2.32}
$$

Now, we use $\tau_D^x$ to denote the first exit time (a stopping time) of the process $X_t$ from the time-space domain $[0,T]\times D$, i.e.,

$$
\tau_D^x = inf\{t>0,(t,X_t)\notin[0,T]\times D\}, \tag{2.33}
$$

where the upper index $x$ means that $X_t$ starts from $x\in D$. Then, we can impose a terminal-boundary condition with $\tau=\tau_D^x$ as follows,

$$
\lim_{t\to\tau_D^x(\omega)} V(t,X_t) = H(\tau_D^x(\omega)) = H_D(x,\omega) \qquad a.s.\ Q^x \tag{2.34}
$$

for all $(t,x)\in[0,T]\times D$. In the case that $H_D(x,\omega)$ is a random variable independent of $x$, the required smooth condition for the well-posedness of the B-SPDE in (2.31) is satisfied. In general, $H_D(x,\omega)$ can be approximated by sufficiently smooth function in $x$ as required.

- **Hall Effects: Stochastic Schrödinger Equation**

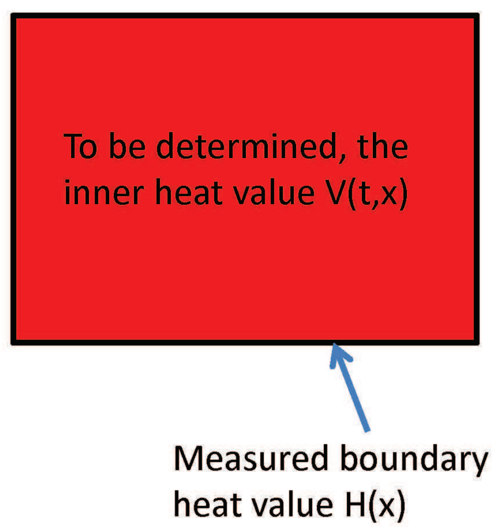


Figure 5: A heat diffusion system by a solution to the Dirichlet-Poisson problem

In quantum physics and statistical mechanics, some phenomena such as Hall/anomalous Hall effects (see, e.g., Hall [25], Karplus and Luttinger [32], and the summarized description at Wikipedia website) are major concerns. By the definition of Hall effect, the movements of quantum particles (e.g., electrons) within a semiconduct/superconduct are along regular paths (see Figure 6 for such an example) if the Lorentz force generated by an external magnetic field with a perpendicular component is imposed. When this phenomenon happens, the collisions of quantum particles will be significantly reduced and the performance of the semiconduct/superconduct will be largely improved. However, in a real application, imposing an external magnetic field is frequently expensive. Thus, people try to develop some magnetic material based semiconduct/superconduct in order that the Hall effect happens naturally (see, e.g., Karplus and Luttinger [32], Chang *et al.* [10]). This phenomenon is called anomalous Hall effect.

Besides observing the Hall/anomalous Hall effects by physical experiments (see, e.g., Hall [25] and Chang *et al.* [10]), one can also analytically study and simulate these effects through a Schrödinger equation (see, e.g., Thouless [51], Chai [8, 9], and the summarized descriptions about density functional theory and time-dependent density functional theory at Wikipedia website). The Schrödinger equation used in most existing studies is a form of the Fokker-Planck's formula (see, e.g., Øksendal [39]). Here, by taking a form of $\mathcal{L}$ in the forward part of the system in (1.1), we can unify these Schrödinger equations (see, e.g., Bouard and Debussche [4, 5], Thouless [51], Chai [8, 9]) into the *generalized stochastic nonlinear*

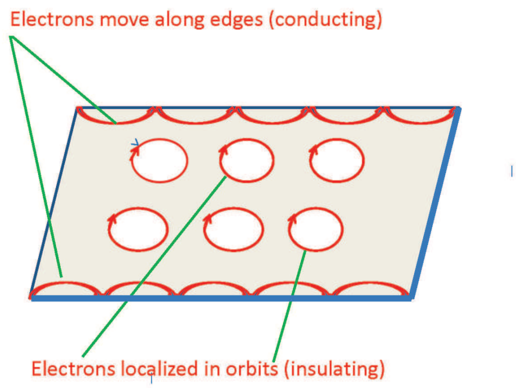


Figure 6: Edge states carry the current

*Schrödinger equation* with absorbing boundaries for each $i \in \{1, ..., 2p\}$),

$$idV(t,x) = \mathcal{L}(t^-,x,V)dt + \mathcal{J}(t^-,x,V)dW(t) + \int_{z>0} \mathcal{I}(t^-,x,V,z)\tilde{N}(\lambda dt, dz), \tag{2.35}$$

where $V$ is single-dimensional (i.e., $q = 1$), $i$ is the imaginary number, and $\mathcal{L}$ is a form of the operator,

$$\mathcal{L}(t,x,V,\cdot) \;=\; \sum_{j=1}^{p} a_{jj}(x)\frac{\partial^2 V(t,x)}{\partial x_j^2} + \sum_{j=1}^{p} b_j(x)\frac{\partial V(t,x)}{\partial x_j} + c(x,V)V(t,x). \tag{2.36}$$

Note that $c(x,V)$ in (2.36) is the potential, which may depend on external temperature and/or external magnetic field. For example, the recent discovery about the Anomalous Hall Effect (see, e.g., Chang *et al.* [10]) is based on a lower temperature and without imposing external magnetic field. Furthermore, the related Schrödinger equation based studies can be found in Chai [8, 9], etc.

Now, if the densities appeared in the Hall/Anomalous Hall Effects are the target stationary distributions (i.e., the terminal values $H(T,x)$ in (1.1) are given), we can take $\bar{\mathcal{L}}$ in the system of (1.1) to be a form of the operator in (2.36). Then, we can find the initial and transient distributions of quantum particles by the backward part of the system in (1.1). From physical viewpoint, this study provides insights about how to characterize and manufacture the magnetic material based semiconductor/superconduct.

## 3 The Coupled FB-SDEs with Lévy Jumps and Skew Reflections

### 3.1 The Coupled FB-SDEs and Its Well-Posedness

In this section, we suppose that the process $X$ governed by the forward SDE in (1.3) lives in a state space $D$ (e.g., a $p$-dimensional positive orthant or a $p$-dimensional rectangle). Furthermore, let $D_i = \{x \in R^p, x \cdot n_i = b_i\}$ for $i \in \{1, ..., b\}$ be the $i$th boundary face of $D$, where $b_i = 0$ for $i \in \{1, ..., p\}$, $b_i$ is some positive constant for $i \in \{p+1, ..., b\}$, and $n_i$ is the inward unit normal vector on the boundary face $D_i$. For convenience, we define $N = (n_1, ..., n_b)$. In addition, let $R$ in (1.3) be a $p \times b$ matrix with $b \in \{p, 2p\}$, whose $i$th column denoted by $p$-dimensional vector $v_i$ is the reflection direction of $X$ on $D_i$. The process $Y$ in (1.3) is a nondecreasing predictable process with $Y(0) = 0$ and boundary regulating property as explained in (1.3). In queueing system, this process is called boundary idle time or blocking process.

Similarly, we assume that $V$ takes value in a region $\bar{D}$ with boundary face $\bar{D}_i = \{v \in R^q, v \cdot \bar{n}_i = \bar{b}_i\}$ for $i \in \{1, ..., \bar{b}\}$, where $\bar{n}_i$ is the inward unit normal vector on the boundary face $\bar{D}_i$. For convenience, we define $\bar{N} = (\bar{n}_1, ..., \bar{n}_{\bar{b}})$. In finance, the given constant $\bar{b}_i$ is called early exercise reward. Furthermore, $S$ in (1.3) is supposed to be a $q \times \bar{b}$ matrix for a known $\bar{b} \in \{q, 2q\}$. In addition, $F(\cdot)$ in (1.3) is a nondecreasing predictable process with $F(0) = 0$ and boundary regulating property as explained in (1.3).

To guarantee the existence and uniqueness of an adapted 6-tuple weak solution to the coupled FB-SDEs in (1.3), we need to introduce the completely-$\mathcal{S}$ condition on the reflection matrix $R$ (and similarly on $S$).

**Definition 3.1** *A $p \times p$ square matrix $R$ is called completely-$\mathcal{S}$ if and only if there is $x > 0$ such that $\tilde{R}x > 0$ for each principal sub-matrix $\tilde{R}$ of $R$, where the vector inequalities are to be interpreted componentwise. Furthermore, a $p \times b$ matrix $R$ is called completely-$\mathcal{S}$ if and only if each $p \times p$ square sub-matrix of $N'R$ is completely-S.*

Note that the completely-$\mathcal{S}$ condition on the reflection matrices guarantees that the coupled FB-SDEs are of inward reflection on each boundary and corner of the orthant or the rectangle (see, e.g., Figure 2 and Dai [14]). Furthermore, the reflection appeared here is called skew reflection that is a generalization of the conventional mirror (or called symmetry) reflection.

Now, the coefficient functions given in (1.3) are assumed to be $\{\mathcal{F}_t\}$-predictable and are detailed as follows,

$$
\begin{aligned}
b(t,x,u) \equiv b(t,x,v,\bar{v},\tilde{v},u,\cdot) &\ :\ [0,T] \times R^p \times R^q \times R^{q\times d} \times R^{q\times h} \times U \to R^p,\\
\sigma(t,x,u) \equiv \sigma(t,x,v,\bar{v},\tilde{v},u,\cdot) &\ :\ [0,T] \times R^p \times R^q \times R^{q\times d} \times R^{q\times h} \times U \to R^{p\times d},\\
\eta(t,x,u) \equiv \eta(t,x,v,\bar{v},\tilde{v},u,z,\cdot) &\ :\ [0,T] \times R^p \times R^q \times R^{q\times d} \times R^{q\times h} \times U \times \mathcal{Z}^h \to R^{p\times h},\\
c(t,x,u) \equiv c(t,x,v,\bar{v},\tilde{v},u,\cdot) &\ :\ [0,T] \times R^p \times R^q \times R^{q\times d} \times R^{q\times h} \times U \to R^q,\\
\alpha(t,x,u) \equiv \sigma(t,x,v,\bar{v},\tilde{v},u,\cdot) &\ :\ [0,T] \times R^p \times R^q \times R^{q\times d} \times R^{q\times h} \times U \to R^{q\times d},\\
\zeta(t,x,u) \equiv \gamma(t,x,v,\bar{v},\tilde{v},u,z,\cdot) &\ :\ [0,T] \times R^p \times R^q \times R^{q\times d} \times R^{q\times h} \times U \times \mathcal{Z}^h \to R^{q\times h}.
\end{aligned}
$$

For $f, f^1, f^2 \in \{b, \sigma, c, \alpha\}$, we suppose that

$$\begin{aligned}
(3.1) \qquad \|f(u)\| &\leq L(t,\omega)\left(1+\|x\|+\|v\|+\|\bar{v}\|+\|\tilde{v}\|_\nu\right), \\
(3.2) \qquad \left\|f^2(u)-f^1(u)\right\| &\leq L(t,\omega)\left(\left\|x^2-x^1\right\|+\left\|v^2-v^1\right\|\right. \\
&\qquad \left.+\left\|\bar{v}^2-\bar{v}^1\right\|+\left\|\tilde{v}^2-\tilde{v}^1\right\|_\nu\right).
\end{aligned}$$

Furthermore, for each $f, f^1, f^2 \in \{\gamma, \zeta\}$ and $z \in \mathcal{Z}^h$, we suppose that

$$\begin{aligned}
(3.3) \qquad &\sum_{i=1}^h \int_{\mathcal{Z}} \|f_i(u,z_i)\|^2 \lambda_i \nu_i(dz_i) \\
&\leq L^2(t,\omega)\left(1+\|x\|^2+\|v\|^2+\|\bar{v}\|^2+\|\tilde{v}\|_\nu^2\right),
\end{aligned}$$

where $f_i$ is the $i$th column of $f$, and

$$\begin{aligned}
(3.4) \qquad &\sum_{i=1}^h \int_{\mathcal{Z}} \left\|f_i^2(u,z_i)-f_i^1(u,z_i)\right\|^2 \lambda_i \nu_i(dz_i) \\
&\leq L^2(t,\omega)\left(\left\|x^2-x^1\right\|^2+\left\|v^2-v^1\right\|^2+\left\|\bar{v}^2-\bar{v}^1\right\|^2+\left\|\tilde{v}^2-\tilde{v}^1\right\|_\nu^2\right).
\end{aligned}$$

In addition, we assume that the terminal value $H(x) \equiv H(x,\cdot)$ satisfies the condition,

$$(3.5) \qquad \|H(x)\| \leq L(t,\omega)(1+\|x\|).$$

Finally, $L$ in (3.1)-(3.4) and (3.5) is assumed to be a known non-negative stochastic process that is $\{\mathcal{F}_t\}$-adapted and mean-squarely integrable, i.e.,

$$(3.6) \qquad E\left[\int_0^T L^2(t)dt\right] < \infty.$$

**Theorem 3.1** *Under conditions (3.1)-(3.6), the following two claims are true:*

1. *If $S$ and $R$ satisfy the completely-$\mathcal{S}$ condition, there exists a unique adapted 6-tuple weak solution to the system in (1.3) when at least one of the forward and backward SDEs has reflection boundary;*

2. *Furthermore, if each $q \times q$ sub-principal matrix of $\bar{N}'S$ and each $p \times p$ sub-principal matrix of $N'R$ are invertible or if both of the SDEs have no reflection boundaries, there is a unique adapted 6-tuple strong solution to the system in (1.3).*

Due to the length, the proof of Theorem 3.1 is postponed to Section 6.

### 3.2 Resolution via Coupled FB-SPDEs

In this subsection, we consider a particular case of the coupled FB-SPDEs in (1.1) but with an additional equation, which corresponds to the special forms of partial differential operators $\bar{\mathcal{L}}$, $\bar{\mathcal{J}}$, and $\bar{\mathcal{I}}$. More precisely, for each $l \in \{0, 1, ..., q\}$, we define

$$\begin{aligned}
(3.7) \qquad & \bar{\mathcal{L}}_l(t, x, U, V, \bar{V}, \tilde{V}, u) \\
\equiv \quad & \sum_{i,j=1}^{p} (\sigma\sigma')_{ij}(t, x, u)\frac{\partial^2 V_l(t, x)}{\partial x_i \partial x_j} + \sum_{i=1}^{p}\left(b_i(t, x, u) + \sum_{j=1}^{b} v_{ij}\gamma_j(t, x)\right)\frac{\partial V_l(t, x)}{\partial x_i} \\
& + \sum_{j=1}^{d}\sum_{i=1}^{p}\sigma_{ji}(t, x, u)\frac{\partial \alpha_{lj}(t, x, u)}{\partial x_i} - c_l(t, x, u) + \sum_{k=1}^{q} s_{lk}\beta_k(t, x) \\
& - \sum_{j=1}^{h}\int_{\mathcal{Z}}\left(V_l(t, x + \eta_j(t, x, u, z_j)) - V_l(t, x) - \sum_{i=1}^{p}\frac{\partial V_l(t, x)}{\partial x_i}\eta_{ij}(t, x, u, z_j)\right)\nu_j(dz_j) \\
& - \sum_{j=1}^{h}\int_{\mathcal{Z}}\left(\tilde{\zeta}_{lj}(t, x + \eta_j(t, x, u, z_j)), u, z_j) - \tilde{\zeta}_{lj}(t, x, u, z_j)\right)\nu_j(dz_j),
\end{aligned}$$

where $\eta_{ij}$ and $\eta_j$ for $i \in \{1, ..., p\}$ and $j \in \{1, ..., h\}$ are the $(i, j)$th entry and the $j$th column of $\eta$ respectively. Furthermore,

$$(3.8) \qquad c_0(t, x, u) = \sum_{l=1}^{q} c_l(t, x, u),$$

$$(3.9) \qquad \tilde{\zeta}_{0j}(t, x, u, z_j)) = \sum_{l=1}^{q} \zeta_{lj}(t, x, u, z_j)),$$

and $\zeta_{lj}$ for $l \in \{1, ..., q\}$ and $j \in \{1, ..., h\}$ is the $(i, j)$th entry of $\zeta$. In addition, $\gamma_j(t, x)$ for $j \in \{1, ..., b\}$ and $\beta_k(t, x)$ for $k \in \{1, ..., q\}$ are some functions in $t$ and $x$.

Note that, the partial derivative

$$\frac{\partial \alpha_{lj}(t, x, u)}{\partial x_i} \text{ for each } l \in \{0, 1, ..., q\},\ i \in \{1, ..., p\},\ j \in \{1, ..., d\}$$

should be interpreted according to chain rule since $\alpha(t, x)$ is also a function in $x$ through $(V, \bar{V}, \tilde{V})(t, x)$ and $u(t, x)$, where

$$(3.10) \qquad \alpha_{0j}(t, x, u) = \sum_{l=1}^{q} \alpha_{lj}(t, x, u).$$

Finally, we define

$$\begin{aligned}
(3.11) \qquad \bar{\mathcal{J}}(t, x, U, V, \bar{V}, \tilde{V}) &= -\bar{V}(t, x), \\
(3.12) \qquad \bar{\mathcal{I}}(t, x, U, V, \bar{V}, \tilde{V}, z) &= -\tilde{V}(t, x), \\
(3.13) \qquad V(T, x) &= H(x),
\end{aligned}$$

where, we assume that $H \in L^2_{\mathcal{F}_T}(\Omega, C^\infty(D; R^q))$. Then, we have the following definition.

**Definition 3.2** *$\mathcal{C}$ is called the admissible set of adapted control processes if $\{\bar{\mathcal{L}}_l(t,x,U,V,\bar{V},\tilde{V},u), l\in\{0,1,...,q\}\}$ together with $\{\mathcal{L},\mathcal{J},\mathcal{I},\bar{\mathcal{J}},\bar{\mathcal{I}}\}$ satisfy the conditions as stated in Theorem 2.1 (or Theorem 2.2).*

**Theorem 3.2** *Let $(U(t,x),V(t,x),\bar{V}(t,x),\tilde{V}(t,x,\cdot))$ be the unique adapted 4-tuple strong solution to the $(r,q+1)$-dimensional coupled FB-SPDEs in (1.1), which corresponds to specific $\{\bar{\mathcal{L}},\bar{\mathcal{J}},\bar{\mathcal{I}}\}$ in (3.7)-(3.12), terminal condition in (3.13), and a control process $u\in\mathcal{C}$. If $S$ and $R$ satisfy the completely-$\mathcal{S}$ condition, the following claims in are true:*

1. *There exists a unique adapted 6-tuple weak solution $((X(t),Y(t)),\ (V(t),\bar{V}(t),\tilde{V}(t,z),F(t)))$ to the system in (1.3) when at least one of the SDEs has reflection boundary, where*

$$V_l(t) = V_l(t,X(t)), \tag{3.14}$$

$$\bar{V}_{lj}(t) = -\left(\alpha_{lj}(t,X(t),u)+\sum_{i=1}^{p}\sigma_{li}(t,X(t),u)\frac{\partial V_l(t,X(t))}{\partial x_i}\right), \tag{3.15}$$

$$\begin{aligned}\tilde{V}_{lj}(t,z)) &= -\left(V_l(t,X(t)+\eta_j(t,X(t),u,z_j))-V_l(t,X(t))\right)\\ &\quad -\zeta_{lj}(t,X(t)+\eta_j(t,X(t),u,z_j),u,z_j)\end{aligned} \tag{3.16}$$

*for $l\in\{0,1,...,q\}$ and $j\in\{1,...,h\}$, where*

$$\alpha(t,X(t),u) = \alpha(t,X(t),V(t,X(t)),\bar{V}(t,X(t)),\tilde{V}(t,X(t),\cdot),u(t,X(t)),\cdot), \tag{3.17}$$

$$\eta(t,X(t),u) = \eta(t,X(t),V(t,X(t)),\bar{V}(t,X(t)),\tilde{V}(t,X(t),\cdot),u(t,X(t)),\cdot), \tag{3.18}$$

$$\zeta(t,X(t),u) = \zeta(t,X(t),V(t,X(t)),\bar{V}(t,X(t)),\tilde{V}(t,X(t),\cdot),u(t,X(t)),\cdot); \tag{3.19}$$

2. *There is a unique adapted 6-tuple strong solution to the system in (1.3) when each $q\times q$ sub-principal matrix of $\bar{N}'S$ and each $p\times p$ sub-principal matrix of $N'R$ are invertible or when both of the SDEs have no reflection boundaries.*

## 4 Connections to Non-Zero-Sum SDGs and Queues

### 4.1 Non-Zero-Sum SDGs

By Theorem 3.2, we suppose that the 4-tuple $(X,V,\bar{V},\tilde{V})$ in (4.1) is part of a solution $(X,Y,V,\bar{V},\tilde{V},F)$ to the non-Markvian system of coupled FB-SDEs with Lévy jumps and skew reflections in (1.3). Then, let $u(\cdot)$ be the corresponding $B$-valued ($B\subset R^q$) and $\{\mathcal{F}_t\}$-adapted control process, whose $l$th component $u_l(\cdot)$ for each $l\in\{1,...,q\}$ is the $l$th player's control policy. Furthermore, we assume that the utility function for each player $l\in\{1,...,q\}$ is defined by

$$\begin{cases} c_l(t,X(t),u) & \equiv c_l(t,X(t),V(t,X(t)),\bar{V}(t,X(t)),\tilde{V}(t,X(t)),u(t,X(t))),\\ c_0(t,X(t),u) & \equiv \sum_{l=1}^{q} c_l(t,X(t),u),\end{cases} \tag{4.1}$$

Thus, it follows from (3.18)-(3.19), (3.9)-(3.10), and (4.1) that the value functions $\{V_l^u(0), l\in\{1,...,q\}\}$ in (1.4) are now well defined. Then, we can introduce the following concepts.

**Definition 4.1** *By a non-zero-sum SDG to the system in (1.3), we mean that each player $l \in \{1, ..., q\}$ chooses an optimal policy to maximize his own value function expressed in (1.4). Furthermore, the value functions $\{V_l^u(0), l \in \{1, ..., q\}\}$ do not have to add up to a constant (e.g., zero), or in other words, the SDG is not necessarily a zero-sum one.*

**Definition 4.2** *$u^*(\cdot)$ is called a Pareto optimal Nash equilibrium policy process if, the process is also an optimal one to the sum of all the q players' value functions at time zero; no player will profit by unilaterally changing his own policy when all the other players' policies keep the same. Mathematically,*

$$V_0^{u^*}(0) \geq V_0^u(0), \quad V_l^{u^*}(0) \geq V_l^{u^*_{-l}}(0) \tag{4.2}$$

*for each $l \in \{0, 1, ..., q\}$ and any given admissible control policy $u$, where*

$$u^*_{-l} = (u_1^*, ..., u_{l-1}^*, u_l, u_{l+1}^*, ..., u_q^*).$$

**Definition 4.3** *$\{\bar{\mathcal{L}}_l(t, x, U, V, \bar{V}, \tilde{V}, u), l \in \{0, 1, ..., q\}\}$ together with $\{\mathcal{L}, \mathcal{J}, \mathcal{I}\}$ are called satisfying the comparison principle in terms of $u$ if, for any two $u^i \in \mathcal{C}$ with $i \in \{1, 2\}$ and any two $\mathcal{F}_T$-measurable $H^i$ with associated two solutions $(U^i, V^i, \bar{V}^i, \tilde{V}^i)(t, x)$, respectively, of (1.3) such that*

$$\begin{aligned} \bar{\mathcal{L}}_l(t, x, U^1, V^1, \bar{V}^1, \tilde{V}^1, u^1) &\leq \bar{\mathcal{L}}_l(t, x, U^2, V^2, \bar{V}^2, \tilde{V}^2, u^2), \\ H^1(x) &\leq H^2(x) \end{aligned}$$

*for all $(t, x) \in [0, T] \times D$, we have*

$$V^1(t, x) \leq V^2(t, x).$$

**Theorem 4.1** *Let $(U(t, x), V(t, x), \bar{V}(t, x), \tilde{V}(t, x, \cdot))$ be the unique adapted 4-tuple strong solution to the $(r, q+1)$-dimensional FB-SPDEs in (1.1), which corresponds to specific $\{\bar{\mathcal{L}}, \bar{\mathcal{J}}, \bar{\mathcal{I}}\}$ in (3.7)-(3.12), terminal condition in (3.13), and a control process $u \in \mathcal{C}$. Suppose that $S$ and $R$ satisfy the completely-$\mathcal{S}$ condition. If $\{\bar{\mathcal{L}}_l(t, x, U, V, \bar{V}, \tilde{V}, u), l \in \{0, 1, ..., q\}\}$ together with $\{\mathcal{L}, \mathcal{J}, \mathcal{I}, \bar{\mathcal{J}}, \bar{\mathcal{I}}\}$ for suitably chosen $\gamma(t, x)$ and $\beta(t, x)$ satisfy the comparison principle in terms of $u$, the following two claims are true:*

1. *There is a Pareto optimal Nash equilibrium point $u^*(t, X(t))$ to the non-zero-sum SDG problem in (1.4) when both of the SDEs in (1.3) have no reflection boundaries and if $\gamma(t, x) = \beta(t, x) \equiv 0$;*

2. *There is an approximated Pareto optimal Nash equilibrium point $u^*(t, X(t))$ to the non-zero-sum SDG problem in (1.4) when at least one of the SDEs in (1.3) has reflection boundary and if $\gamma(t, x)$, $\beta(t, x)$ are taken to be infinitely smooth approximated functions of $\frac{dF}{dt}(t,x)$ and $\frac{dY}{dt}(t, x)$ in $x$.*

### 4.2 Queues and Reflecting Diffusions

Queueing networks widely appear in real-world applications such as those in service, cloud computing, and communication systems. They typically consist of arrival processes, service processes, and buffer storages with certain kind of service regime and network architecture (see, e.g., an example with $p$-job classes in Figure 4). The major performance measure for this system is the *queue length process* denoted by $Q(\cdot) = (Q_1(\cdot), ..., Q_p(\cdot))'$, where $Q_i(t)$ is the number of $i$th class jobs stored in the $i$th buffer for each $i \in \{1, ..., p\}$ at time $t$. Let $Q(0)$ be the initial queue length for the system. Then, the queueing dynamics of the system can be presented by

$$Q(t) = Q(0) + A(t) - D(t), \tag{4.3}$$

where, the $i$th component $A_i(t)$ of $A(t)$ for each $i \in \{1, ..., p\}$ is the total number of jobs arrived to buffer $i$ by time $t$, and the $i$th component $D_i(t)$ of $D(t)$ is the total number of jobs departed from buffer $i$ by time $t$. In the following discussions, we use two generalized ways to characterize the arrival and departure processes.

First, we assume that each $A_i(\cdot)$ for $i \in \{1, ..., p\}$ is a time-inhomogeneous Lévy process with intensity measure $a_i(t, Q(t), z_i)dt\nu_i(dz_i)$ that is the job arrival rate to buffer $i$ at time $t$ and depends on the queue state at time $t$. Similarly, we assume that each $D_i(\cdot)$ is also a time-inhomogeneous Lévy process with intensity measure $d_i(t, Q(t), z_i)dt\nu_i(dz_i)$ that is the assigned service rate to buffer $i$ at time $t$. Furthermore, we assume that the routing proportion from buffer $j$ to buffer $i$ for jobs finishing service at buffer $j$ is $p_{ji}(t, Q(t), z_j)$. Then, by the F-SDE in (1.3) and the discussions in Applebaum [1], the queue length process in (4.3) for this case can be further expressed by

$$\begin{aligned} dQ_i(t) \;=\; & \Bigg(\int_{\mathcal{Z}} a_i(t, Q(t), z_i)\nu_i(dz_i) \\ & + \sum_{j\neq i}\int_{\mathcal{Z}} p_{ji}(t, Q(t), z_j)d_j(t, Q(t), z_j)I_{\{Q_j(t)>0\}}\nu_j(dz_j) \\ & - \int_{\mathcal{Z}} d_i(t, Q(t), z_i)I_{\{Q_i(t)>0\}}\nu_i(dz_i)\Bigg)\, dt \\ & + \int_{\mathcal{Z}} a_i(t, Q(t), z_i)\tilde{N}_i(dt, dz_i) \\ & + \sum_{j\neq i}\int_{\mathcal{Z}} p_{ji}(t, Q(t), z_j)d_j(t, Q(t), z_j)I_{\{Q_j(t)>0\}}\tilde{N}_j(dt, dz_j) \\ & - \int_{\mathcal{Z}} d_i(t, Q(t), z_i)I_{\{Q_i(t)>0\}}\tilde{N}_i(dt, dz_i) \\ & + \sum_{j=1}^{b} R_{ij}(t, Q(t))dY_j(t), \end{aligned} \tag{4.4}$$

where, $\mathcal{Z} = R_+$, $Y_j(t)$ in (4.4) for each $j \in \{1, ..., b\}$ is the Skorohod regulator process and it can increase only at time $t$ when $Q_j(t) = 0$. Note that $R(t, Q(t))$ is a reflection matrix that

may be time and queue state dependent, and the coefficients in (4.4) may be discontinuous at the queue state $Q_i(t) = 0$. However, since the system in (4.4) is designed in a controllable manner, the service rate $d_i(s, Q(s))$ can always be set to be zero when $Q_i(t) = 0$, which implies that the reflection part in (4.4) can be removed. Hence, the generalized Lipschitz and linear growth conditions in (3.1)-(3.4) may be reasonably imposed to the system in (4.4). Thus, the system derived in (4.4) can be well-posed. Furthermore, the optimal policies in terms of cost, profit, and system performance can be designed and analyzed (see, e.g., the related illustration in the coming Subsection 4.3). Interested readers can also find some specific formulations of the queueing system (4.4) in Mandelbaum and Massey [36], Mandelbaum and Pats [37], and Konstantopoulos *et al.* [33], etc.

Second, we assume that both the arrival and service processes are described by renewal processes, renewal reward processes, or doubly stochastic renewal processes. In this case, the driven processes for the queueing system do not have the nice statistical properties such as memoryless and stationary increment ones. Thus, it is usually impossible to conduct exact analysis concerning the distribution of $Q(\cdot)$. However, under certain conditions (e.g., the arrival rates close to the associated service rates), one can show that the corresponding sequence of diffusion-scaled queue length processes converges in distribution to a $p$-dimensional reflecting Brownian motion (RBM) (see, e.g., Dai [14], Dai and Dai [12], Dai and Jiang [21]), or more generally, a reflecting diffusion with regime switching (RDRS) (see, e.g., Dai [18]). In other words, we have that

$$\hat{Q}^r(\cdot) \equiv \frac{1}{r} Q(r^2 \cdot) \Rightarrow \hat{Q}(\cdot) \quad \text{along} \quad r \in \{1, 2, ...\}, \tag{4.5}$$

where "$\Rightarrow$" means "converges in distribution" and $\hat{Q}(\cdot)$ is a RBM or a RDRS.

To be simple, we consider the case that the limit $\hat{Q}(\cdot)$ in (4.5) is a RBM living in the state space $D$ introduced in Section 3. Furthermore, let $\theta$ be a vector in $R^p$ and $\Gamma$ be a $p \times p$ symmetric and positive definite matrix. Then, we can introduce the definition of a RBM (see, e.g, Dai [14]) as follows.

**Definition 4.4** *A semimartingale RBM associated with the data* $(\mathbf{S}, \theta, \Gamma, R)$ *that has initial distribution* $\pi$ *is a continuous,* $\{\mathcal{F}_t\}$*-adapted, p-dimensional process* $Z$ *defined on some filtered probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$ *such that under* $\mathbf{P}$*,*

$$X(t) = Z(t) + RY(t) \quad \textit{for all } t \geq 0,$$

*where*

1. *$X$ has continuous paths in $\mathbf{S}$, $\mathbf{P}$-a.s.,*

2. *under $\mathbf{P}$, $Z$ is a $p$-dimensional Brownian motion with drift vector $\theta$ and covariance matrix $\Gamma$ such that $\{Z(t) - \theta t, \mathcal{F}_t, t \geq 0\}$ is a martingale and $PZ^{-1}(0) = \pi$,*

3. *$Y$ is a $\{\mathcal{F}_t\}$-adapted, $b$-dimensional process such that $\mathbf{P}$-a.s., for each $i \in \{1, ..., b\}$, the $i$th component $Y_i$ of $Y$ satisfies*

*(a)* $Y_i(0) = 0$*,*

*(b)* $Y_i$ *is continuous and non-decreasing,*

*(c)* $Y_i$ *can increase only when* $Z$ *is on the face* $D_i$*, i.e., as given in (1.3).*

From the physical viewpoint of queueing system (see, e.g., Dai [14, 18]) and the discussion in Reiman and Williams [47], the pushing process $Y$ in Definition 4.4 can be assumed to a.s. satisfy

$$Y_i(t) = \int_0^t I_{D_i}(X(s))ds. \tag{4.6}$$

Now, assume that $H(x)$ is the stationary distribution that we expect for the RBM $X$. For example, in reality, it is the given distribution of the long-run average queue lengths among different users or job classes. Theoretically, it can be computed by a method (e.g., the finite element method designed and implemented in Dai *et al.* [14, 49]). Then, we can use a B-PDE or a B-SPDE (a special form of the system in (1.1)) to get the transition function at each time point to reach the targeted or limiting stationary distribution $H(x)$ for the RBM $X$ for a given initial distribution (e.g., $X(0) = 0$ a.s. in many situations). Hence, the corresponding performance measures of the physical queueing system can be estimated. More precisely, we have the following theorem and related remark.

**Theorem 4.2** *Suppose that the reflection matrix satisfies the completely-$\mathcal{S}$ condition. Then, the transition function of the RBM* $X$ *over* $[0, T]$ *is determined by*

$$V(t,x) \;\; = \;\; H(x) + \int_t^T \mathcal{L}(s,x,V)ds, \tag{4.7}$$

*where* $V$ *is a* 1*-dimensional function. Furthermore,* $\mathcal{L}$ *is the following form of partial differential operator*

$$\mathcal{L}(t,x,V,\cdot) \;\; = \;\; (\mathcal{K}(t,x,V,\cdot), \mathcal{D}_1(t,x,V,\cdot), ..., \mathcal{D}_b(t,x,V,\cdot)), \tag{4.8}$$

$$\mathcal{K}(t,x,V,\cdot) \;\; = \;\; \sum_{i,j=1}^{p} \Gamma_{ij} \frac{\partial^2 V(t,x)}{\partial x_i \partial x_j} + \theta \cdot \bigtriangledown V(t,x) + \sum_{i=1}^{b} \mathcal{D}_i(t,x,V,\cdot), \tag{4.9}$$

$$\mathcal{D}_i(t,x,V,\cdot) \;\; = \;\; (v_i \cdot \bigtriangledown V(t,x)) I_{D_i}(x) \;\; \text{for} \;\; x \in D_i \;\; \text{with} \;\; i \in \{1,...,b\}, \tag{4.10}$$

*where* $\bigtriangledown V$ *is the gradient vector of* $V$ *in* $x$ *and* $I_{F_i}$ *is the indicator function over the set* $F_i$*.*

Proof. It follows from the completely-$\mathcal{S}$ condition that the RBM $X$ is a strong Markov process (see, e.g., Dai and Williams [13]). Then, by applying the Itô's formula (see, e.g., Protter [46]) and Fokker-Planck's formula (or called Kolmogorov's forward/backward equations, see, e.g., Øksendal [39]), we know that the claim stated in the theorem is true. □

**Remark 4.1** *Owing to the uncertainty error of measurement,* $H(x)$ *could be random. Furthermore, the coefficients in (4.9) may also be random, e.g., for the case that the limit* $\hat{Q}(\cdot)$ *is a RDRS. Thus, a B-SPDE can be introduced. Furthermore, the indicator function* $I_{F_i}(x)$ *can be approximated by a sufficient smooth function in order to apply Theorem 2.1 to the equation in (4.7), which is reasonable from the viewpoint of numerical computation.*

### 4.3 Queueing Based Game Problem

From the information system displayed in Figures 3-4 (presenting a parallel-server queueing system with $q = p$), we can give an explanation about the decision process for such a game problem. In this game, each player (or called user in Dai [18]) relates to a control process $u_l(\cdot)$ for $l \in \{1, ..., q\}$ over certain resource pool (e.g., called the transmission rate allocation process over a randomly evolving capacity region in Dai [18]). In the meanwhile, each player $l$ is assigned a surrogate utility function $c_l$ of his submitted bid (called queue length in Dai [18], or the approximated queue length RBM $Z$ in Definition 4.4) to the network, the price from the network to him, and the control policy at each time point by the central information administrative. Then, an optimal and/or fair control process can be determined by the utility functions of all players, queueing process, and the available resource constraint in a cooperative way (see, e.g., Jones [29]).

## 5 Proofs of Theorem 2.1 and Theorem 2.2

We justify the two theorems by first proving three lemmas in the following subsection.

### 5.1 The Lemmas

**Lemma 5.1** *Assume that the conditions in Theorem 2.1 hold and take a quadruplet for each fixed $x \in D$ and $z \in \mathcal{Z}^h$,*

$$(U^1(\cdot,x), V^1(\cdot,x), \bar{V}^1(\cdot,x), \tilde{V}^1(\cdot,x,z)) \in \mathcal{Q}^2_{\mathcal{F}}([0,T] \times D). \tag{5.1}$$

*Then, there exists another quadruplet $(U^2(\cdot,x), V^2(\cdot,x), \bar{V}^2(\cdot,x), \tilde{V}^2(\cdot,x,z))$ such that*

$$\begin{cases} U^2(t,x) & = G(x) + \int_0^t \mathcal{L}(s^-, x, U^1, V^1, \bar{V}^1, \tilde{V}^1)ds \\ & \quad + \int_0^t \mathcal{J}(s^-, x, U^1, V^1, \bar{V}^1, \tilde{V}^1)dW(s) \\ & \quad + \int_0^t \int_{\mathcal{Z}^h} \mathcal{I}(s^-, x, U^1, V^1, \bar{V}^1, \tilde{V}^1, z)\tilde{N}(\lambda ds, dz), \\ V^2(t,x) & = H(x) + \int_t^T \bar{\mathcal{L}}(s^-, x, U^1, V^1, \bar{V}^1, \tilde{V}^1)ds \\ & \quad + \int_t^T \Big(\bar{\mathcal{J}}(s^-, x, U^1, V^1, \bar{V}^1, \tilde{V}^1) \\ & \qquad + \bar{V}^1(s^-, x) - \bar{V}^2(s^-, x)\Big) dW(s) \\ & \quad + \int_t^T \int_{\mathcal{Z}^h} \Big(\bar{\mathcal{I}}(s^-, x, U^1, V^1, \bar{V}^1, \tilde{V}^1, z) \\ & \qquad + \tilde{V}^1(s^-, x, z) - \tilde{V}^2(s^-, x, z)\Big) \tilde{N}(\lambda ds, dz), \end{cases} \tag{5.2}$$

*where $(U^2, V^2)$ is a $\{\mathcal{F}_t\}$-adapted càdlàg process and $(\bar{V}^2, \tilde{V}^2)$ is the corresponding predictable process. Furthermore, for each $x \in D$,*

$$E\left[\int_0^T \|U^2(t,x)\|^2 dt\right] < \infty, \tag{5.3}$$

$$E\left[\int_0^T \|V^2(t,x)\|^2 dt\right] < \infty, \tag{5.4}$$

$$E\left[\int_0^T \|\bar{V}^2(t,x)\|^2 dt\right] < \infty, \tag{5.5}$$

$$E\left[\sum_{i=1}^{h}\int_0^T \int_{\mathcal{Z}} \left\|\tilde{V}_i^2(t,x,z_i)\right\|^2 \nu_i(dz_i)dt\right] < \infty. \tag{5.6}$$

Proof. For each fixed $x \in D$ and a quadruplet as stated in (5.1), it follows from conditions (2.17)-(2.26) that

$$\mathcal{L}(\cdot,x,U^1,V^1,\bar{V}^1,\tilde{V}^1) \in L^2_{\mathcal{F}}([0,T],C^\infty(D,R^r)), \tag{5.7}$$

$$\mathcal{J}(\cdot,x,U^1,V^1,\bar{V}^1,\tilde{V}^1) \in L^2_{\mathcal{F}}([0,T],C^\infty(D,R^{r\times d})), \tag{5.8}$$

$$\mathcal{I}(\cdot,x,U^1,V^1,\bar{V}^1,\tilde{V}^1) \in L^2_{\mathcal{F}}([0,T]\times\mathcal{Z}^h,C^\infty(D,R^{r\times h})), \tag{5.9}$$

$$\bar{\mathcal{L}}(\cdot,x,U^1,V^1,\bar{V}^1,\tilde{V}^1) \in L^2_{\mathcal{F}}([0,T],C^\infty(D,R^q)), \tag{5.10}$$

$$\bar{\mathcal{J}}(\cdot,x,U^1,V^1,\bar{V}^1,\tilde{V}^1) \in L^2_{\mathcal{F}}([0,T],C^\infty(D,R^{q\times d})), \tag{5.11}$$

$$\bar{\mathcal{I}}(\cdot,x,U^1,V^1,\bar{V}^1,\tilde{V}^1) \in L^2_{\mathcal{F}}([0,T]\times\mathcal{Z}^h,C^\infty(D,R^{q\times h})). \tag{5.12}$$

By considering $\mathcal{L}$, $\mathcal{J}$, and $\mathcal{I}$ in (5.7)-(5.9) as new starting $\mathcal{L}(\cdot,x,0,0,0,0)$, $\mathcal{J}(\cdot,x,0,0,0,0)$, and $\mathcal{I}(\cdot,x,0,0,0,0)$, we can define $U^2$ by the forward iteration in (5.2). Furthermore, $U^2$ is a $\{\mathcal{F}_t\}$-adapted càdlàg process that is square-integrable for each $x \in D$ in the sense of (5.3).

Now, consider $\bar{\mathcal{L}}$, $\bar{\mathcal{J}}$, and $\bar{\mathcal{I}}$ in (5.10)-(5.12) as new starting $\bar{\mathcal{L}}(\cdot,x,0,0,0,0)$, $\bar{\mathcal{J}}(\cdot,x,0,0,0,0)$, and $\bar{\mathcal{I}}(\cdot,x,0,0,0,0)$. Then, it follows from the Martingale representation theorem (see, e.g., Theorem 5.3.5 in page 266 of Applebaum [1]) that there are unique predictable processes $\bar{V}^2(\cdot,x)$ and $\tilde{V}^2(\cdot,x,z)$ such that

$$\begin{aligned}
&\hat{V}^2(t,x) \\
\equiv\ & E\Bigg[H(x)+\int_0^T \bar{\mathcal{L}}(s^-,x,U^1,V^1,\bar{V}^1,\tilde{V}^1)ds \\
&+\int_0^T \left(\bar{\mathcal{J}}(s^-,x,U^1,V^1,\bar{V}^1,\tilde{V}^1)+\bar{V}^1(s^-,x)\right)dW(s) \\
&+\int_0^T\int_{\mathcal{Z}} \left(\bar{\mathcal{I}}(s^-,x,U^1,V^1,\bar{V}^1,\tilde{V}^1,z)+\tilde{V}^1(s^-,x,z)\right)\tilde{N}(\lambda ds,dz)\Bigg|\mathcal{F}_t\Bigg] \\
=\ & \hat{V}^2(0,x)+\int_0^t \bar{V}^2(s^-,x)dW(s)+\int_0^t\int_{\mathcal{Z}}\tilde{V}^2(s^-,x,z)\tilde{N}(\lambda ds,dz).
\end{aligned} \tag{5.13}$$

Furthermore, $\bar{V}^2$ and $\tilde{V}^2$ are square-integrable for each $x \in D$ in the sense of (5.5)-(5.6), and

$$\begin{aligned}
&\hat{V}^2(0,x) \\
=\ & \hat{V}^2(T,x)-\int_0^T \bar{V}^2(s^-,x)dW(s)-\int_0^T\int_{\mathcal{Z}}\tilde{V}^2(s^-,x,z)\tilde{N}(\lambda ds,dz) \\
=\ & H(x)+\int_0^T \bar{\mathcal{L}}(s^-,x,U^1,V^1,\bar{V}^1,\tilde{V}^1)ds \\
&+\int_0^T \left(\bar{\mathcal{J}}(s^-,x,U^1,V^1,\bar{V}^1,\tilde{V}^1)+\bar{V}^1(s^-,x)-\bar{V}^2(s^-,x)\right)dW(s)
\end{aligned} \tag{5.14}$$

$$+\int_0^T\int_{\mathcal{Z}}\left(\bar{\mathcal{I}}(s^-,x,U^1,V^1,\bar{V}^1,\tilde{V}^1,z)+\tilde{V}^1(s^-,x,z)-\tilde{V}^2(s^-,x,z)\right)\tilde{N}(\lambda ds,dz).$$

Owing to the corollary in page 8 of Protter [46], $\hat{V}^2(\cdot,x)$ can be taken as a càdlàg process. Now, define a process $V^2$ given by

$$\begin{aligned}(5.15)\quad V^2(t,x) &= E\left[H(x)+\int_t^T\bar{\mathcal{L}}(s^-,x,U^1,V^1,\bar{V}^1,\tilde{V}^1)ds\right.\\ &+\int_t^T\left(\bar{\mathcal{J}}(s^-,x,U^1,V^1,\bar{V}^1,\tilde{V}^1)+\bar{V}^1(s^-,x)\right)dW(s)\\ &\left.+\int_t^T\int_{\mathcal{Z}}\left(\bar{\mathcal{I}}(s^-,x,U^1,V^1,z)+\tilde{V}^1(s^-,x,z)\right)\tilde{N}(\lambda ds,dz)\right|\mathcal{F}_t\right].\end{aligned}$$

Thus, it follows from (2.19)-(2.20) and simple calculation that $V^2(\cdot,x)$ is square-integrable in the sense of (5.4). In addition, by (5.13)-(5.15), we know that

$$\begin{aligned}(5.16)\qquad V^2(t,x) &= \hat{V}^2(t,x)-\int_0^t\bar{\mathcal{L}}(s^-,x,U^1,V^1,\bar{V}^1,\tilde{V}^1)ds\\ &-\int_0^t\left(\bar{\mathcal{J}}(s^-,x,U^1,V^1,\bar{V}^1,\tilde{V}^1)+\bar{V}^1(s^-,x)\right)dW(s)\\ &-\int_0^t\int_{\mathcal{Z}}\left(\bar{\mathcal{I}}(s^-,x,U^1,V^1,z)+\tilde{V}^1(s^-,x,z)\right)\tilde{N}(\lambda ds,dz),\end{aligned}$$

which implies that $V^2(\cdot,x)$ is a càdlàg process.

Hence, for a given quadruplet in (5.1), it follows from (5.13)-(5.14) and (5.16) that the associated quadruplet $(U^2(\cdot,x),\ V^2(\cdot,x),\ \bar{V}^2(\cdot,x),\ \tilde{V}^2(\cdot,x,z))$ satisfies the equation (5.2) as stated in the lemma. Furthermore, we know that

$$\begin{aligned}(5.17)\qquad & V^2(t,x)\\ \equiv\ & V^2(0,x)-\int_0^t\bar{\mathcal{L}}(s^-,x,U^1,V^1,\bar{V}^1,\tilde{V}^2)ds\\ &-\int_0^t\left(\bar{\mathcal{J}}(s^-,x,U^1,V^1,\bar{V}^1,\tilde{V}^1)+\bar{V}^1(s^-,x)-\bar{V}^2(s^-,x)\right)dW(s)\\ &-\int_0^t\int_{\mathcal{Z}}\left(\bar{\mathcal{I}}(s^-,x,U^1,V^1,\bar{V}^-,\tilde{V}^1,z)+\tilde{V}^1(s^-,x,z)-\tilde{V}^2(s^-,x,z)\right)\tilde{N}(\lambda ds,dz).\end{aligned}$$

Thus, we complete the proof of Lemma 5.1. □

**Lemma 5.2** *Under the conditions of Theorem 2.1, consider a quadruplet as in (5.1) for each fixed $x\in D$ and $z\in\mathcal{Z}^h$. Define $(U(t,x),V(t,x),\bar{V}(t,x),\tilde{V}(t,x,z))$ by (5.2). Then,*

*$(U^{(c)}(\cdot,x),\ V^{(c)}(\cdot,x),\ \bar{V}^{(c)}(\cdot,x),\ \tilde{V}^{(c)}(\cdot,x,z))$ for each $c\in\{0,1,...,\}$ exists a.s. and satisfies*

$$(5.18)\qquad \begin{cases} U^{(c)}_{i_1\ldots i_p}(t,x) &= G^{(c)}_{i_1\ldots i_p}(x)+\int_0^t\mathcal{L}^{(c)}_{i_1\ldots i_p}(s^-,x,U^1,V^1,\bar{V}^1,\tilde{V}^1)ds\\ &\quad+\int_0^t\mathcal{J}^{(c)}_{i_1\ldots i_p}(s^-,x,U^1,V^1,\bar{V}^1,\tilde{V}^1)dW(s)\\ &\quad+\int_0^t\int_{\mathcal{Z}}\mathcal{I}^{(c)}_{i_1\ldots i_p}(s^-,x,U^1,V^1,\bar{V}^1,\tilde{V}^1,z)\tilde{N}(\lambda ds,dz),\\ V^{(c)}_{i_1\ldots i_p}(t,x) &= H^{(c)}_{i_1\ldots i_p}(x)+\int_t^T\bar{\mathcal{L}}^{(c)}_{i_1\ldots i_p}(s^-,x,U^1,V^1,\bar{V}^1,\tilde{V}^1)ds\\ &\quad+\int_t^T\Big(\bar{\mathcal{J}}^{(c)}_{i_1\ldots i_p}(s^-,x,U^1,V^1,\bar{V}^1,\tilde{V}^1)\\ &\qquad+\bar{V}^{1,(c)}_{i_1\ldots i_p}(s^-,x)-\bar{V}^{(c)}_{i_1\ldots i_p}(s^-,x)\Big)dW(s)\\ &\quad+\int_t^T\int_{\mathcal{Z}}\Big(\bar{\mathcal{I}}^{(c)}_{i_1\ldots i_p}(s^-,x,U^1,V^1,\bar{V}^1,\tilde{V}^1,z)\\ &\qquad+\tilde{V}^{1,(c)}_{i_1\ldots i_p}(s^-,x,z)-\tilde{V}^{(c)}_{i_1\ldots i_p}(s^-,x,z)\Big)\tilde{N}(\lambda ds,dz), \end{cases}$$

*where $i_1+...+i_p=c$ and $i_l\in\{0,1,...,c\}$ with $l\in\{1,...,p\}$. Furthermore, $(U^{(c)}_{i_1\ldots i_p},\ V^{(c)}_{i_1\ldots i_p})$ for each $c\in\{0,1,...\}$ is a $\{\mathcal{F}_t\}$-adapted càdlàg process and $(\bar{V}^{(c)}_{i_1\ldots i_p},\ \tilde{V}^{(c)}_{i_1\ldots i_p})$ is the associated predictable processes. All of them are squarely-integrable in the senses of (5.4)-(5.6).*

PROOF. Without loss of generality, we only consider the point $x\in D$, which is an interior one of $D$. Otherwise, we can use the corresponding derivative in a one-side manner to replace the one in the following proof.

First, we show that the claim in the lemma is true for $c=1$. To do so, for each given $t\in[0,T], x\in D, z\in\mathcal{Z}^h$, and $(U^1(t,x),V^1(t,x),\bar{V}^1(t,x),\tilde{V}^1(t,x,z))$ as in the lemma, let

$$(5.19)\qquad (U^{(1)}_{i_l}(t,x),V^{(1)}_{i_l}(t,x),\bar{V}^{(1)}_{i_l}(t,x),\tilde{V}^{(1)}_{i_l}(t,x,z))$$

be defined by (5.2) but each $\mathcal{A}\in\{\mathcal{L},\mathcal{J},\mathcal{I},\bar{\mathcal{L}},\bar{\mathcal{J}},\bar{\mathcal{I}}\}$ is replaced by its first-order partial derivative

$$\mathcal{A}^{(1)}_{i_l}\in\left\{\mathcal{L}^{(1)}_{i_l},\mathcal{J}^{(1)}_{i_l},\mathcal{I}^{(1)}_{i_l},\bar{\mathcal{L}}^{(1)}_{i_l},\bar{\mathcal{J}}^{(1)}_{i_l},\bar{\mathcal{I}}^{(1)}_{i_l}\right\}$$

with respect to $x_l$ for $l\in\{1,...,p\}$ if $i_l=1$. Then, we can show that the quadruplet defined in (5.19) for each $l$ is the required first-order partial derivative of $(U,V,\bar{V},\tilde{V})$ in (5.2) for the given $(U^1,V^1,\bar{V}^1,\tilde{V}^1)$.

In fact, considering an interior point $x$ of $D$, we can take sufficiently small constant $\delta$ such that $x+\delta e_l\in D$, where $e_l$ is the unit vector whose $l$th component is one and others are zero. Without loss of generality, we assume that $\delta>0$. Then, for each $f\in\{U,V,\bar{V},\tilde{V},U^1,V^1,\bar{V}^1,\tilde{V}^1\}$ and $i_l=1$ with $l\in\{1,...,p\}$, we define

$$(5.20)\qquad f_{i_l,\delta}(t,x)\equiv f(t,x+\delta e_l).$$

Furthermore, let

$$(5.21)\qquad \Delta f^{(1)}_{i_l,\delta}(t,x)=\frac{f_{i_l,\delta}(t,x)-f(t,x)}{\delta}-f^{(1)}_{i_l}(t,x),$$

and let

$$
\begin{aligned}
(5.22)\qquad & \Delta\mathcal{A}^{(1)}_{i_l,\delta}(t,x,U^1,V^1,\bar{V}^1,\tilde{V}^1) \\
= \;& \frac{1}{\delta}\Big(\mathcal{A}(t,x+\delta e_l,U^1(t,x+\delta e_l),V^1(t,x+\delta e_l),\bar{V}^1(t,x+\delta e_l),\tilde{V}^1(t,x+\delta e_l,z)) \\
& \quad -\mathcal{A}(t,x,U^1(s,x),V^1(t,x),\bar{V}^1(t,x),\tilde{V}^1(t,x,z))\Big) \\
& -\mathcal{A}^{(1)}_{i_l}(t,x,U^1(s,x),V^1(t,x),\bar{V}^1(t,x),\tilde{V}^1(t,x,z))
\end{aligned}
$$

for each $\mathcal{A}\in\{\mathcal{L},\mathcal{J},\mathcal{I},\bar{\mathcal{L}},\bar{\mathcal{J}},\bar{\mathcal{I}}\}$.

Now, let $\mathrm{Tr}(A)$ denote the trace of the matrix $A'A$ for a given matrix $A$ and let $(\mathrm{Tr}(A))_j$ be the $j$th term in the summation of the trace. Furthermore, for each fixed $t\in[0,T]$, $\delta>0$, and $\gamma>0$, define

$$
\begin{aligned}
(5.23)\qquad Z_\delta(t,x) &\equiv \zeta(\Delta U^{(1)}_{i_l,\delta}(t,x)+\Delta V^{(1)}_{i_l,\delta}(t,x)) \\
&= \left(\mathrm{Tr}\left(\Delta U^{(1)}_{i_l,\delta}(t,x)\right)+\mathrm{Tr}\left(\Delta V^{(1)}_{i_l,\delta}(t,x)\right)\right)e^{2\gamma t}.
\end{aligned}
$$

Then, it follows from (5.17) and the Itô's formula (see, e.g., Theorem 1.14 and Theorem 1.16 in pages 6-9 of Øksendal and Sulem [40]) that

$$
\begin{aligned}
(5.24)\quad & Z_\delta(t,x)+\int_t^T \mathrm{Tr}\Big(\Delta\bar{\mathcal{J}}^{(1)}_{i_l,\delta}(s^-,x,U^1,V^1,\bar{V}^1,\tilde{V}^1) \\
& \qquad +\Delta\bar{V}^{1,(1)}_{i_l,\delta}(s^-,x)-\Delta\bar{V}^{(1)}_{i_l,\delta}(s,x)\Big)e^{2\gamma s}ds \\
& +\sum_{j=1}^h\int_t^T\int_{\mathcal{Z}}\Big(\mathrm{Tr}\Big(\Delta\bar{\mathcal{I}}^{(1)}_{i_l,\delta}(s^-,x,U^1,V^1,\bar{V}^1,\tilde{V}^1,z) \\
& \qquad +\Delta\tilde{V}^{1,(1)}_{i_l,\delta}(s^-,x,z_j)-\Delta\tilde{V}^{(1)}_{i_l,\delta}(s^-,x,z)\Big)\Big)_j e^{2\gamma s}N_j(\lambda_j ds,dz_j) \\
= \;& 2\int_0^t\left(-\gamma\mathrm{Tr}\left(\Delta U^{(1)}_{i_l,\delta}(s,x)\right)+\left(\Delta U^{(1)}_{i_l,\delta}(s,x)\right)'\left(\Delta\mathcal{L}^{(1)}_{i_l,\delta}(s,x,U^1,V^1,\bar{V}^1,\tilde{V}^1)\right)\right)e^{2\gamma s}ds \\
& +2\int_t^T\left(-\gamma\mathrm{Tr}\left(\Delta V^{(1)}_{i_l,\delta}(s,x)\right)+\left(\Delta V^{(1)}_{i_l,\delta}(s,x)\right)'\left(\Delta\bar{\mathcal{L}}^{(1)}_{i_l,\delta}(s,x,U^1,V^1,\bar{V}^1,\tilde{V}^1)\right)\right)e^{2\gamma s}ds \\
& -M_\delta(t,x) \\
\le \;& \left(-2\gamma+\frac{1}{\hat{\gamma}}\right)\left(\int_0^t\mathrm{Tr}\left(\Delta U^{(1)}_{i_l,\delta}(s,x)\right)e^{2\gamma s}ds+\int_t^T\mathrm{Tr}\left(\Delta V^{(1)}_{i_l,\delta}(s,x)\right)e^{2\gamma s}ds\right) \\
& +\hat{\gamma}\int_0^t\left\|\Delta\mathcal{L}^{(1)}_{i_l,\delta}(s,x,U^1,V^1,\bar{V}^1,\tilde{V}^1)\right\|^2e^{2\gamma s}ds \\
& +\hat{\gamma}\int_t^T\left\|\Delta\bar{\mathcal{L}}^{(1)}_{i_l,\delta}(s,x,U^1,V^1,\bar{V}^1,\tilde{V}^1)\right\|^2e^{2\gamma s}ds \\
& -M_\delta(t,x) \\
= \;& \hat{\gamma}\int_0^t\left\|\Delta\mathcal{L}^{(1)}_{i_l,\delta}(s,x,U^1,V^1,\bar{V}^1,\tilde{V}^1)\right\|^2e^{2\gamma s}ds
\end{aligned}
$$

$$
\begin{aligned}
&+\hat{\gamma}\int_t^T \left\|\Delta\bar{\mathcal{L}}^{(1)}_{i_l,\delta}(s,x,U^1,V^1,\bar{V}^1,\tilde{V}^1)\right\|^2 e^{2\gamma s}ds \\
&-M_\delta(t,x)
\end{aligned}
$$

if, in the last equality, we take

$$
\hat{\gamma}=\frac{1}{2\gamma}>0. \tag{5.25}
$$

Note that $M_\delta(t,x)$ in (5.24) is a martingale of the form,

$$
\begin{aligned}
&M_\delta(t,x) \\
= &-2\sum_{j=1}^{d}\int_0^t \left(\Delta U^{(1)}_{i_l,\delta}(s^-,x)\right)'\Delta(\mathcal{J}_j)^{(1)}_{i_l,\delta}(s^-,x,U^1,V^1,\bar{V}^1,\tilde{V}^1)e^{2\gamma s}dW_j(s) \\
&-2\sum_{j=1}^{h}\int_0^t\int_{\mathcal{Z}} \left(\Delta U^{(1)}_{i_l,\delta}(s^-,x)\right)'\Delta(\mathcal{I}_j)^{(1)}_{i_l,\delta}(s^-,x,U^1,V^1,\bar{V}^1,\tilde{V}^1,z_j)e^{2\gamma s}\tilde{N}_j(\lambda_j ds,dz_j) \\
&+2\sum_{j=1}^{d}\int_t^T \left(\Delta V^{(1)}_{i_l,\delta}(s^-,x)\right)'\left(\Delta(\bar{\mathcal{J}}_j)^{(1)}_{i_l,\delta}(s^-,x,U^1,V^1,\bar{V}^1,\tilde{V}^1)\right. \\
&\qquad\left.+\Delta(\bar{V}^1_j)^{(1)}_{i_l,\delta}(s^-,x)-\Delta(\bar{V}_j)^{(1)}_{i_l,\delta}(s^-,x)\right)e^{2\gamma s}dW_j(s) \\
&+2\sum_{j=1}^{h}\int_t^T\int_{\mathcal{Z}} \left(\Delta V^{(1)}_{i_l,\delta}(s^-,x)\right)'\left(\Delta(\bar{\mathcal{I}}_j)^{(1)}_{i_l,\delta}(s^-,x,U^1,V^1,\bar{V}^1,\tilde{V}^1,z_j)\right. \\
&\qquad\left.+\Delta(\tilde{V}^1_j)^{(1)}_{i_l,\delta}(s^-,x,z_j)-\Delta(\tilde{V}_j)^{(1)}_{i_l,\delta}(s^-,x,z_j)\right)e^{2\gamma s}\tilde{N}_j(\lambda_j ds,dz_j).
\end{aligned} \tag{5.26}
$$

Thus, by the martingale property and (5.24), we know that

$$
\begin{aligned}
&E\left[Z_\delta(t,x)+\int_t^T \mathrm{Tr}\left(\Delta\bar{\mathcal{J}}^{(1)}_{i_l,\delta}(s^-,x,U^1,V^1,\bar{V}^1,\tilde{V}^1)\right.\right. \\
&\qquad\left.+\Delta\bar{V}^{1,(1)}_{i_l,\delta}(s^-,x)-\Delta\bar{V}^{(1)}_{i_l,\delta}(s,x)\right)e^{2\gamma s}ds \\
&\quad+\sum_{j=1}^{h}\int_t^T\int_{\mathcal{Z}}\left(\mathrm{Tr}\left(\Delta\bar{\mathcal{I}}^{(1)}_{i_l,\delta}(s^-,x,U^1,V^1,\bar{V}^1,\tilde{V}^1,z)\right.\right. \\
&\qquad\left.\left.\left.+\Delta\tilde{V}^{1,(1)}_{i_l,\delta}(s^-,x,z)-\Delta\tilde{V}^{(1)}_{i_l,\delta}(s^-,x,z)\right)\right)_j e^{2\gamma s}N_j(\lambda_j ds,dz_j)\right] \\
\leq\;&\hat{\gamma}E\left[\int_0^t \left\|\Delta\mathcal{L}^{(1)}_{i_l,\delta}(s,x,U^1,V^1,\bar{V}^1,\tilde{V}^1)\right\|^2 e^{2\gamma s}ds\right] \\
&+\hat{\gamma}\left[\int_t^T \left\|\Delta\bar{\mathcal{L}}^{(1)}_{i_l,\delta}(s,x,U^1,V^1,\bar{V}^1,\tilde{V}^1)\right\|^2 e^{2\gamma s}ds\right].
\end{aligned} \tag{5.27}
$$

Furthermore, by (5.24)-(5.27) and the Burkholder-Davis-Gundy's inequality (see, e.g., Theorem 48 in page 193 of Protter [46]), we have the following observation,

$$
E\left[\sup_{0\leq t\leq T}|M_\delta(t,x)|\right] \tag{5.28}
$$

$$\le \hat{\gamma} K_1 E\left[\int_0^t \left\|\Delta \mathcal{L}^{(1)}_{i_l,\delta}(s,x,U^1,V^1,\bar{V}^1,\tilde{V}^1)\right\|^2 e^{2\gamma s}ds\right]$$
$$+\hat{\gamma} K_1 \left[\int_t^T \left\|\Delta \bar{\mathcal{L}}^{(1)}_{i_l,\delta}(s,x,U^1,V^1,\bar{V}^1,\tilde{V}^1)\right\|^2 e^{2\gamma s}ds\right],$$

where $K_1$ is some nonnegative constant depending only on $K_{D,0}$, $K_{D,1}$, $T$, and $d$. Note that, the detailed estimation procedure for the quantity on the right-hand side of (5.28) is postponed to the same argument used for (5.55) in the proof of Lemma 5.3 since more exact calculations are required there.

Next, for each fixed $t \in [0,T]$, $x \in D$, and $\sigma > 0$, consider the random variable set $\{Z_\delta(t,x),\ \delta \in [0,\sigma]\}$. It follows from Lemma 1.3 in pages 6-7 of Peskir and Shiryaev [45] that there is a countable subset $\mathcal{C} = \{\delta_1, \delta_2, ...\} \subset [0,\sigma]$ such that

$$\text{esssup}_{\delta\in[0,\sigma]} Z_\delta(t,x) = \sup_{\delta\in\mathcal{C}} Z_\delta(t,x), \quad \text{a.s.,} \tag{5.29}$$

where "esssup" denotes the essential supremum. Furthermore, take

$$\begin{cases} \bar{Z}_{\delta_1}(t,x) = Z_{\delta_1}(t,x), \\ \bar{Z}_{\delta_{n+1}}(t,x) = \bar{Z}_{\delta_n}(t,x) \vee Z_{\delta_{n+1}}(t,x) \ \text{ for } \ n \in \{1,2,...\}, \end{cases} \tag{5.30}$$

where $\alpha \vee \beta = \max\{\alpha,\beta\}$ for any two real numbers $\alpha$ and $\beta$. Obviously,

$$\begin{cases} Z_\delta(t,x) \le \bar{Z}_\delta(t,x) & \text{for each } \ \delta \in \mathcal{C} \\ \bar{Z}_{\delta_1}(t,x) \le \bar{Z}_{\delta_2}(t,x) & \text{for any } \ \delta_1,\delta_2 \in \mathcal{C} \ \text{ satisfying } \ \delta_1 \le \delta_2. \end{cases} \tag{5.31}$$

The second inequality in (5.31) implies that the set $\left\{\bar{Z}_\delta(t,x), \delta \in \mathcal{C}\right\}$ is upwards directed. Hence, for each $t \in [0,T]$, $x \in D$, $\sigma > 0$, and the associated sequence of $\{\delta_n, n = 1,2,...\}$, it follows from (5.29) that

$$\begin{aligned} & E\left[\text{esssup}_{0\le\delta\le\sigma} Z_\delta(t,x)\right] \\ \le\ & E\left[\text{esssup}_{\delta\in\mathcal{C}} \bar{Z}_\delta(t,x)\right] \\ =\ & \lim_{n\to\infty} E\left[\bar{Z}_{\delta_n}(t,x)\right] \\ =\ & \lim_{n\to\infty} E\left[\max_{\delta\in\{\delta_1,...,\delta_n\}} Z_\delta(t,x)\right]. \end{aligned} \tag{5.32}$$

In addition, for each fixed $n \in \{2,3,...\}$, let

$$\bar{M}_{\delta_n}(t,x) = M_{\delta_n}(t,x) I_{\{Z_{\delta_n} \ge \bar{Z}_{\delta_{n-1}}\}} + M_{\delta_{n-1}}(t,x) I_{\{Z_{\delta_n} < \bar{Z}_{\delta_{n-1}}\}}. \tag{5.33}$$

Thus, by the induction method in terms of $n \in \{1,2,...\}$ and (5.24), we know that

$$\begin{aligned} & E\left[\max_{\delta\in\{\delta_1,...,\delta_n\}} Z_\delta(t,x)\right] \\ \le\ & \hat{\gamma} \lim_{n\to\infty} E\left[\int_0^t \max_{\delta\in\{\delta_1,...,\delta_n\}} \left\|\Delta \mathcal{L}^{(1)}_{i_l,\delta}(s,x,U^1,V^1,\bar{V}^1,\tilde{V}^1)\right\|^2 e^{2\gamma s}ds\right. \end{aligned} \tag{5.34}$$

$$
\begin{aligned}
& + \int_t^T \max_{\delta\in\{\delta_1,\ldots,\delta_n\}} \left\|\Delta\bar{\mathcal{L}}^{(1)}_{i_l,\delta}(s,x,U^1,V^1,\bar{V}^1,\tilde{V}^1)\right\|^2 e^{2\gamma s}ds\Big] \\
& - \lim_{n\to\infty} E\left[\bar{M}_{\delta_n}(t,x)\right] \\
\leq\ & KE\Big[\int_0^t \mathrm{esssup}_{0\leq\delta\leq\sigma}\left\|\Delta\mathcal{L}^{(1)}_{i_l,\delta}(s,x,U^1,V^1,\bar{V}^1,\tilde{V}^1)\right\|^2 e^{2\gamma s}ds \\
& \quad + \int_t^T \mathrm{esssup}_{0\leq\delta\leq\sigma}\left\|\Delta\bar{\mathcal{L}}^{(1)}_{i_l,\delta}(s,x,U^1,V^1,\bar{V}^1,\tilde{V}^1)\right\|^2 e^{2\gamma s}ds\Big] \\
& \quad + \int_0^T \mathrm{esssup}_{0\leq\delta\leq\sigma}\left\|\Delta\mathcal{J}^{(1)}_{i_l,\delta}(s^-,x,U^1,V^1,\bar{V}^1,\tilde{V}^1)\right\|^2 e^{2\gamma s}ds \\
& \quad + \int_0^T \sum_{i=1}^h \int_{\mathcal{Z}} \mathrm{esssup}_{0\leq\delta\leq\sigma}\left\|\Delta\mathcal{I}^{(1)}_{i,i_l,\delta}(s^-,x,U^1,V^1,\bar{V}^1,\tilde{V}^1,z_i)\right\|^2 e^{2\gamma s}\lambda_i\nu_i(dz_i)ds\Big],
\end{aligned}
$$

where $K$ is a nonnegative constant depending only on $K_{D,0}$, $d$, $T$, and $\gamma$. Note that, in the second inequality, we have used the fact in (5.27) and the following observation

$$
\left|E\left[\bar{M}_{\delta_n}(t,x)\right]\right| \leq E\left[\sup_{t\in[0,T]}\left\|M_{\delta_n}(t,x)\right\|\right] + E\left[\sup_{t\in[0,T]}\left\|M_{\delta_{n-1}}(t,x)\right\|\right]. \tag{5.35}
$$

Now, recall the condition that

$$
(U^1(\cdot,x),V^1(\cdot,x),\bar{V}^1(\cdot,x),\tilde{V}^1(\cdot,x,z)) \in \mathcal{Q}^2_{\mathcal{F}}([0,T]\times D).
$$

Then, for each $x\in D$, $z\in\mathcal{Z}^h$, any $c\in\{0,1,...\}$, and any small number $\xi$ such that $x+\xi e_l\in D$, we have that

$$
\begin{aligned}
& \left\|(U^{1,(c)}(t,x+\xi e_l),V^{1,(c)}(t,x+\xi e_l),\bar{V}^{1,(c)}(t,x+\xi e_l),\tilde{V}^{1,(c)}(t,x+\xi e_l,z))\right\| \\
\leq\ & \left\|\left(\max_{x\in D}\left\|U^{1,(c)}(t,x)\right\|,\max_{x\in D}\left\|V^{1,(c)}(t,x)\right\|,\max_{x\in D}\left\|\bar{V}^{1,(c)}(t,x)\right\|,\max_{x\in D}\left\|\tilde{V}^{1,(c)}(t,x,z)\right\|\right)\right\|.
\end{aligned} \tag{5.36}
$$

Note that the related quantities on the right-hand side of (5.36) are squarely integrable a.s. in term of the Lebesgue measure and/or the Lévy measure. Therefore, $\tilde{V}^1(t,x,\cdot)$ (the integration of $\tilde{V}^1(t,x,z)$ with respect to the Lévy measure) is also infinitely smooth in each $x\in D$ due to the Lebesgue's dominated convergence theorem. Thus, by the mean-value theorem, there exist some constants $\xi_1\in(0,\delta)$ and $\xi\in(0,\xi_1)$, which depend on $\delta$, such that

$$
\begin{aligned}
& \Delta\mathcal{A}^{(1)}_{i_l,\delta}(t,x,U^1,V^1,\bar{V}^1,\tilde{V}^1) \\
=\ & \xi_1\mathcal{A}^{(2)}_{i_l}(t,x+\xi e_l,U^1(t,x+\xi e_l),V^1(t,x+\xi e_l),\bar{V}^1(t,x+\xi e_l),\tilde{V}^1(t,x+\xi e_l,\cdot))
\end{aligned} \tag{5.37}
$$

a.s. for each $\mathcal{A}\in\{\mathcal{L},\mathcal{J},\bar{\mathcal{L}}\}$. Due to (5.37), (2.17), and (5.36), the quantity on the left-hand side of (5.37) for all $\delta$ is dominated by a squarely-integrable random variable in terms of the product measure $dt\times dP$. Similarly, for $\mathcal{A}=\bar{\mathcal{J}}$ and each $z\in\mathcal{Z}^h$, we a.s. have that

$$
\begin{aligned}
& \Delta\mathcal{A}^{(1)}_{i_l,\delta}(t,x,U^1,V^1,\bar{V}^1,\tilde{V}^1,z) \\
=\ & \xi_1\mathcal{A}^{(2)}_{i_l}(t,x+\xi e_l,U^1(t,x+\xi e_l),V^1(t,x+\xi e_l),\bar{V}^1(t,x+\xi e_l),\tilde{V}^1(t,x+\xi e_l,z),z).
\end{aligned} \tag{5.38}
$$

Owing to (5.37), (2.18), and (5.36), the quantity on the left-hand side of (5.38) for all $\delta$ is dominated by a squarely-integrable random variable in terms of the product measure $dt \times \nu(dz) \times dP$. Therefore, it follows from (5.32)-(5.34) and the Lebesgue's dominated convergence theorem that

$$
\begin{aligned}
(5.39)\ & \lim_{\sigma\to 0} E\left[\operatorname{esssup}_{0\leq\delta\leq\sigma} Z_\delta(t,x)\right] \\
\leq\ & KE\left[\int_0^t \lim_{\sigma\to 0} \operatorname{esssup}_{0\leq\delta\leq\sigma} \left\|\Delta\mathcal{L}^{(1)}_{i_l,\delta}(s,x,U^1,V^1,\bar{V}^1,\tilde{V}^1)\right\|^2 e^{2\gamma s}ds\right. \\
&+\int_t^T \lim_{\sigma\to 0} \operatorname{esssup}_{0\leq\delta\leq\sigma} \left\|\Delta\bar{\mathcal{L}}^{(1)}_{i_l,\delta}(s,x,U^1,V^1,\bar{V}^1,\tilde{V}^1)\right\|^2 e^{2\gamma s}ds \\
&+\int_0^T \lim_{\sigma\to 0} \operatorname{esssup}_{0\leq\delta\leq\sigma} \left\|\Delta\mathcal{J}^{(1)}_{i_l,\delta}(s^-,x,U^1,V^1,\bar{V}^1,\tilde{V}^1)\right\| e^{2\gamma s}ds \\
&\left.+\int_0^T \sum_{i=1}^h \int_{\mathcal{Z}} \lim_{\sigma\to 0} \operatorname{esssup}_{0\leq\delta\leq\sigma} \left\|\Delta\mathcal{I}^{(1)}_{i_l,\delta}(s^-,x,U^1,V^1,\bar{V}^1,\tilde{V}^1,z_i)\right\| e^{2\gamma s}\lambda_i\nu_i(dz_i)ds\right].
\end{aligned}
$$

Hence, by (5.39) and the Fatou's lemma, we know that, for any sequence $\sigma_n$ satisfying $\sigma_n \to 0$ along $n \in \mathcal{N}$, there is a subsequence $\mathcal{N}' \subset \mathcal{N}$ such that

$$
\operatorname{esssup}_{0\leq\delta\leq\sigma_n} Z_\delta(t,x)) \to 0 \ \text{ along } \ n\in\mathcal{N}' \ \text{ a.s.} \tag{5.40}
$$

The convergence in (5.40) implies that the first-order derivatives of $U$ and $V$ in terms of $x_l$ for each $l \in \{1,...,p\}$ exists. More exactly, they equal $U^{(1)}_{i_l}(t,x)$ and $V^{(1)}_{i_l}(t,x)$ a.s. respectively for each $t\in[0,T]$ and $x \in D$. Furthermore, they are $\{\mathcal{F}_t\}$-adapted.

Now, we prove the claim for $\bar{V}$. In fact, it follows from the proof as in (5.32)-(5.34) that

$$
\begin{aligned}
\lim_{\sigma\to 0} E\left[\int_t^T \operatorname{esssup}_{0\leq\delta\leq\sigma} \operatorname{Tr}\left(\Delta\bar{\mathcal{J}}^{(1)}_{i_l,\delta}(s,x,U^1,V^1,\bar{V}^1,\tilde{V}^1)\right.\right. \\
\left.\left.+\Delta(\bar{V}^1)^{(1)}_{i_l,\delta}(s,x) - \Delta\bar{V}^{(1)}_{i_l,\delta}(s,x)\right) e^{2\gamma s}ds\right]
\end{aligned} \tag{5.41}
$$

is also bounded by the quantity on the right-hand side of (5.39). Thus, by (5.40) and (5.41), we know that

$$
\begin{aligned}
& \lim_{\delta\to 0} \Delta\bar{V}^{(1)}_{i_l,\delta}(t,x) \\
=\ & \lim_{\delta\to 0}\left(\Delta\bar{\mathcal{J}}^{(1)}_{i_l,\delta}(t,x,U^1,V^1,\bar{V}^1,\tilde{V}^1) + \Delta(\bar{V}^1)^{(1)}_{i_l,\delta}(t,x)\right) \\
=\ & 0, \quad \text{a.s.}
\end{aligned}
$$

Hence, the first-order derivative of $\bar{V}$ in $x_l$ for each $l \in \{1,...,p\}$ exists and equals $\bar{V}^{(1)}_{i_l}(t,x)$ a.s. for every $t\in[0,T]$ and $x\in D$. Furthermore, it is a $\{\mathcal{F}_t\}$-predictable process. Similarly, we can get the conclusion for $\tilde{V}^{(1)}_{i_l}(t,x,z)$ associated with each $l$, $t$, $x$, and $z$.

Second, we suppose that $(U^{(c-1)}(t,x), V^{(c-1)}(t,x), \bar{V}^{(c-1)}(t,x), \tilde{V}^{(c-1)}(t,x,z))$ corresponding to a given $(U^1(t,x)$, $V^1(t,x)$, $\bar{V}^1(t,x)$, $\tilde{V}^1(t,x,z)) \in \mathcal{Q}^2_{\mathcal{F}}([0,T]\times D)$ exists for any given $c \in \{1,2,...\}$. Then, we can show that

$$
\left(U^{(c)}(t,x), V^{(c)}(t,x), \bar{V}^{(c)}(t,x), \tilde{V}^{(c)}(t,x,z)\right) \tag{5.42}
$$

exists for the given $c \in \{1, 2, ...\}$.

In fact, consider any fixed nonnegative integer numbers $i_1, ..., i_p$ satisfying $i_1 + ... + i_p = c-1$ for the given $c \in \{1, 2, ...\}$. Take $f \in \{U, V, \bar{V}, \tilde{V}\}$, $l \in \{1, ..., p\}$, and sufficiently small $\delta > 0$. Then, let

$$f^{(c-1)}_{i_1...(i_l+1)...i_p,\delta}(t, x) \equiv f^{(c-1)}_{i_1...i_p}(t, x + \delta e_l) \tag{5.43}$$

correspond to the $(c-1)$th-order partial derivative $\mathcal{A}^{(c-1)}_{i_1...i_p}(s, x + \delta e_l, U^1(s, x + \delta e_l), V^1(s, x + \delta e_l))$ of $\mathcal{A} \in \{\mathcal{L}, \mathcal{J}, \mathcal{I}, \bar{\mathcal{L}}, \bar{\mathcal{J}}, \bar{\mathcal{I}}\}$ via (5.2). Similarly, let

$$(U^{(c)}_{i_1...(i_l+1)...i_p}(t, x), V^{(c)}_{i_1...(i_l+1)...i_p}(t, x), \bar{V}^{(c)}_{i_1...(i_l+1)...i_p}(t, x), \tilde{V}^{(c)}_{i_1...(i_l+1)...i_p}(t, x, z))$$

be defined by (5.2), where $\mathcal{A} \in \{\mathcal{L}, \mathcal{J}, \mathcal{I}, \bar{\mathcal{L}}, \bar{\mathcal{J}}, \bar{\mathcal{I}}\}$ are replaced by their $c$th-order partial derivatives $\mathcal{A}^{(c)}_{i_1...(i_l+1)...i_p}$ corresponding to a given $t, x$, $U^1(t, x)$, $V^1(t, x)$, $\bar{V}^1(t, x)$, $\tilde{V}^1(t, x, z)$. Furthermore, let

$$\Delta f^{(c)}_{i_1...(i_l+1)...i_p,\delta}(t, x) = \frac{f^{(c-1)}_{i_1...(i_l+1)...i_p,\delta}(t, x) - f^{(c-1)}_{i_1...i_p}(t, x)}{\delta} - f^{(c)}_{i_1...(i_l+1)...i_p}(t, x) \tag{5.44}$$

for each $f \in \{U, V, \bar{V}, \tilde{V}, U^1, V^1, \bar{V}^1, \tilde{V}^1\}$. Then, define

$$\begin{aligned} & \Delta\mathcal{A}^{(c)}_{i_1...(i_l+1)...i_p,\delta}(t, x, U^1, V^1) \\ \equiv\ & \frac{1}{\delta}\Big(\mathcal{A}^{(c-1)}_{i_1...i_p}(t, x + \delta e_l, U^1(t, x + \delta e_l) - V^1(t, x + \delta e_l), \cdot) \\ & \quad -\mathcal{A}^{(c-1)}_{i_1...i_p}(s, x, U^1(s, x), V^1(s, x)\cdot)\Big) \\ & -\mathcal{A}^{(c)}_{i_1...(i_l+1)...i_p}(s, x, U^1(s, x), V^1(s, x)\cdot) \end{aligned} \tag{5.45}$$

for each $\mathcal{A} \in \{\mathcal{L}, \mathcal{J}, \mathcal{I}, \bar{\mathcal{L}}, \bar{\mathcal{J}}, \bar{\mathcal{I}}\}$. Thus, by the It$\hat{o}$'s formula and repeating the procedure as used in the first step, we know that

$$(U^{(c)}_{i_1...(i_l+1)...i_p}(t, x), V^{(c)}_{i_1...(i_l+1)...i_p}(t, x), \bar{V}^{(c)}_{i_1...(i_l+1)...i_p}(t, x), \tilde{V}^{(c)}_{i_1...(i_l+1)...i_p}(t, x, z))$$

exist for the given $c \in \{1, 2, ...\}$ and all $l \in \{1, ..., p\}$. Therefore, the claim in (5.42) is true.

Third, by the induction method with respect to $c \in \{1, 2, ...\}$ and the continuity of all partial derivatives in terms of $x \in D$, we know that the claims in the lemma are true. Hence, we finish the proof of Lemma 5.2. □

To state and prove the next lemma, let $D^2_{\mathcal{F}}([0, T], C^\infty(D, R^l))$ with $l \in \{r, q\}$ be the set of $R^l$-valued $\{\mathcal{F}_t\}$-adapted and squarely integrable càdlàg processes as in (2.8). Furthermore, for any given number sequence $\gamma = \{\gamma_c, c = 0, 1, 2, ...\}$ with $\gamma_c \in R$, define $\mathcal{M}^D_\gamma[0, T]$ to be the following Banach space (see, e.g., the related explanation in Yong and Zhou [52], and Situ [50])

$$\begin{aligned} \mathcal{M}^D_\gamma[0, T] \equiv\ & D^2_{\mathcal{F}}([0, T], C^\infty(D, R^r)) \\ & \times D^2_{\mathcal{F}}([0, T], C^\infty(D, R^q)) \\ & \times L^2_{\mathcal{F},p}([0, T], C^\infty(D, R^{q\times d})) \\ & \times L^2_p([0, T] \times R^h_+, C^\infty(D, R^{q\times h})), \end{aligned} \tag{5.46}$$

which is endowed with the norm

$$\left\|(U,V,\bar{V},\tilde{V})\right\|^2_{\mathcal{M}^D_\gamma} \equiv \sum_{c=0}^{\infty}\xi(c)\left\|(U,V,\bar{V},\tilde{V})\right\|^2_{\mathcal{M}^D_{\gamma_c,c}} \tag{5.47}$$

for any given $(U,V,\bar{V},\tilde{V})\in\mathcal{M}^D_\gamma[0,T]$, and

$$\begin{aligned}\left\|(U,V,\bar{V},\tilde{V})\right\|^2_{\mathcal{M}^D_{\gamma_c,c}} &= E\left[\sup_{0\leq t\leq T}\|U(t)\|^2_{C^c(D,q)}e^{2\gamma_c t}\right]\\ &\quad +E\left[\sup_{0\leq t\leq T}\|V(t)\|^2_{C^c(D,q)}e^{2\gamma_c t}\right]\\ &\quad +E\left[\int_0^T\|\bar{V}(t)\|^2_{C^c(D,qd)}e^{2\gamma_c t}dt\right]\\ &\quad +E\left[\int_0^T\left\|\tilde{V}(t)\right\|^2_{\nu,c}e^{2\gamma_c t}dt\right].\end{aligned} \tag{5.48}$$

Then, we have the following lemma.

**Lemma 5.3** *Under the conditions of Theorem 2.1, all the claims in the theorem are true.*

Proof. By (5.2), we can define the following map

$$\Xi:\ (U^1(\cdot,x),V^1(\cdot,x),\bar{V}^1(\cdot,x),\tilde{V}^1(\cdot,x,z))\to(U(\cdot,x),V(\cdot,x),\bar{V}(\cdot,x),\tilde{V}(\cdot,x,z)).$$

Then, we show that $\Xi$ forms a contraction mapping in $\mathcal{M}^D_\gamma[0,T]$. In fact, consider

$$(U^i(\cdot,x),V^i(\cdot,x),\bar{V}^i(\cdot,x),\tilde{V}^i(\cdot,x,z))\in\mathcal{M}^D_\gamma[0,T]$$

for each $i\in\{1,2,...\}$, satisfying

$$\begin{aligned}&(U^{i+1}(\cdot,x),V^{i+1}(\cdot,x),\bar{V}^{i+1}(\cdot,x),\tilde{V}^{i+1}(\cdot,x,z))\\ =\ &\Xi(U^i(\cdot,x),V^i(\cdot,x),\bar{V}^i(\cdot,x),\tilde{V}^i(\cdot,x,z)).\end{aligned}$$

Furthermore, define

$$\Delta f^i=f^{i+1}-f^i\quad\text{with}\quad f\in\{U,V,\bar{V},\tilde{V}\}$$

and take

$$\zeta(\Delta U^i(t,x)+\Delta V^i(t,x))=\left(\mathrm{Tr}\left(\Delta U^i(t,x)\right)+\mathrm{Tr}\left(\Delta V^i(t,x)\right)\right)e^{2\gamma_0 t}. \tag{5.49}$$

Thus, it follows from (2.17) and the similar argument as used in proving (5.24) that, for a $\gamma_0>0$ and each $i\in\{2,3,...\}$,

$$\zeta(\Delta U^i(t,x)+\Delta V^i(t,x)) \tag{5.50}$$

$$
\begin{aligned}
& + \int_t^T \mathrm{Tr}\Big(\Delta\bar{\mathcal{J}}(s,x,U^i,V^i,\bar{V}^i,\tilde{V}^i,U^{i-1},V^{i-1},\bar{V}^{i-1},\tilde{V}^{i-1}) \\
& \qquad\qquad + \Delta\bar{V}^{i-1}(s,x) - \Delta\bar{V}^i(s,x)\Big) e^{2\gamma_0 s} ds \\
& + \sum_{j=1}^h \int_t^T \int_{\mathcal{Z}} \Big(\mathrm{Tr}\Big(\Delta\bar{\mathcal{I}}(s^-,x,U^i,V^i,\bar{V}^i,\tilde{V}^i,U^{i-1},V^{i-1},\bar{V}^{i-1},\tilde{V}^{i-1},z) \\
& \qquad\qquad + \Delta\tilde{V}^{i-1}(s^-,x,z) - \Delta\tilde{V}^i(s^-,x,z)\Big)\Big)_j e^{2\gamma_0 s} N_j(\lambda_j ds, dz_j) \\
\leq\ & \hat{\gamma}_0 \Big(\int_0^t \left\|\Delta\mathcal{L}(s,x,U^i,V^i,\bar{V}^i,\tilde{V}^i,U^{i-1},V^{i-1},\bar{V}^{i-1},\tilde{V}^{i-1})\right\|^2 e^{2\gamma_0 s} ds \\
& + \int_t^T \left\|\Delta\bar{\mathcal{L}}(s,x,U^i,V^i,\bar{V}^i,\tilde{V}^i,U^{i-1},V^{i-1},\bar{V}^{i-1},\tilde{V}^{i-1})\right\|^2 e^{2\gamma_0 s} ds\Big) \\
& - M^i(t,x) \\
\leq\ & \hat{\gamma}_0 K_{a,0} N^{i-1}(t) - M^i(t,x),
\end{aligned}
$$

where $K_{a,0}$ is some nonnegative constant depending only on $K_{D,0}$. For the last inequality in (5.50), we have taken

$$
\hat{\gamma}_0 = \frac{1}{2\gamma_0} > 0. \tag{5.51}
$$

Furthermore, $N^{i-1}(t)$ appeared in (5.50) is given by

$$
\begin{aligned}
& N^{i-1}(t) \\
=\ & \int_0^t \left\|\Delta U^{i-1}(s)\right\|^2_{C^k(D,r)} e^{2\gamma_0 s} ds \\
& + \int_t^T \Big(\left\|\Delta V^{i-1}(s)\right\|^2_{C^k(D,q)} + \left\|\Delta\bar{V}^{i-1}(s)\right\|^2_{C^k(D,qd)} + \left\|\Delta\tilde{V}^{i-1}(s)\right\|^2_{\nu,k}\Big) e^{2\gamma_0 s} ds.
\end{aligned} \tag{5.52}
$$

In addition, $M^i(t,x)$ in (5.50) is a martingale of the form,

$$
\begin{aligned}
& M^i(t,x) = \\
& -2\sum_{j=1}^d \int_0^t \left(\Delta U^i(s^-,x)\right)' \\
& \qquad \Delta\mathcal{J}_j(s^-,x,U^i,V^i,\bar{V}^i,\tilde{V}^i,U^{i-1},V^{i-1},\bar{V}^{i-1},\tilde{V}^{i-1}) e^{2\gamma s} dW_j(s) \\
& -2\sum_{j=1}^h \int_0^t \int_{\mathcal{Z}} \left(\Delta U^i(s^-,x)\right)' \\
& \qquad \Delta\mathcal{I}_j(s^-,x,U^i,V^i,\bar{V}^i,\tilde{V}^i,U^{i-1},V^{i-1},\bar{V}^{i-1},\tilde{V}^{i-1},z_j) e^{2\gamma s} \tilde{N}_j(\lambda_j ds, dz_j) \\
& +2\sum_{j=1}^d \int_t^T \left(\Delta V^i(s^-,x)\right)' \Big(\Delta\bar{\mathcal{J}}_j(s^-,x,U^i,V^i,\bar{V}^i,\tilde{V}^i,U^{i-1},V^{i-1},\bar{V}^{i-1},\tilde{V}^{i-1}) \\
& \qquad + (\Delta\bar{V}^{i-1})_j(s^-,x) - (\Delta\bar{V}^i)_j(s^-,x)\Big) e^{2\gamma_0 s} dW_j(s)
\end{aligned} \tag{5.53}
$$

$$+2\sum_{j=1}^{h}\int_t^T\int_{\mathcal{Z}}\left((\Delta V^i)_j(s^-,x)\right)'\Big(\Delta\bar{\mathcal{I}}_j(s^-,x,U^i,V^i,\bar{V}^i,\tilde{V}^i,U^{i-1},V^{i-1},\bar{V}^{i-1},\tilde{V}^{i-1},z_j)$$
$$+(\Delta\tilde{V}^{i-1})_j(s^-,x,z_j)-(\Delta\tilde{V}^i)_j(s^-,x,z_j)\Big)e^{2\gamma_0 s}\tilde{N}_j(\lambda_j ds,dz_j).$$

Then, it follows from (5.50)-(5.53) and the martingale properties related to the It$\hat{o}$'s stochastic integral that

$$\begin{aligned}
(5.54)\qquad & E\Big[\left(\zeta(\Delta U^i(t,x)+\Delta V^i(t,x))\right)e^{2\gamma_0 t}\\
&+\int_t^T \mathrm{Tr}\Big(\Delta\bar{\mathcal{J}}(s,x,U^i,V^i,\bar{V}^i,\tilde{V}^i,U^{i-1},V^{i-1},\bar{V}^{i-1},\tilde{V}^{i-1})\\
&\qquad +\Delta\bar{V}^{i-1}(s,x)-\Delta\bar{V}^i(s,x)\Big)e^{2\gamma_0 s}ds\\
&+\sum_{j=1}^{h}\int_t^T\int_{\mathcal{Z}}\Big(\mathrm{Tr}\Big(\Delta\mathcal{I}(s,x,U^i,V^i,\bar{V}^i,\tilde{V}^i,U^{i-1},V^{i-1},\bar{V}^{i-1},\tilde{V}^{i-1},z)\\
&\qquad +\Delta\tilde{V}^{i-1}(s^-,x,z)-\Delta\tilde{V}^i(s^-,x,z)\Big)\Big)_j e^{2\gamma_0 s}\lambda_j ds\nu_j(dz_j)\Big]\\
\leq\ & \hat{\gamma}_0(T+1)K_{a,0}\left\|(\Delta U^{i-1},V^{i-1},\Delta\bar{V}^{i-1},\Delta\tilde{V}^{i-1})\right\|^2_{\mathcal{M}^D_{\gamma_0,k}}.
\end{aligned}$$

Next, it follows from (5.53) that

$$\begin{aligned}
(5.55)\qquad & E\left[\sup_{0\leq t\leq T}\left|M^i(t,x)\right|\right]\\
\leq\ & 2\sum_{j=1}^{d}E\Bigg[\sup_{0\leq t\leq T}\Big|\int_0^t\left(\Delta U^i(s^-,x)\right)'\\
&\qquad \Delta\mathcal{J}_j(s^-,x,U^i,V^i,\bar{V}^i,\tilde{V}^i,U^{i-1},V^{i-1},\bar{V}^{i-1},\tilde{V}^{i-1})e^{2\gamma_0 s}dW_j(s)\Big|\Bigg]\\
&+2\sum_{j=1}^{h}E\Bigg[\sup_{0\leq t\leq T}\Big|\int_0^t\int_{\mathcal{Z}}\left(\Delta U^i(s^-,x)\right)'\\
&\qquad \Delta\mathcal{I}_j(s^-,x,U^i,V^i,\bar{V}^i,\tilde{V}^i,U^{i-1},V^{i-1},\bar{V}^{i-1},\tilde{V}^{i-1},z_j)e^{2\gamma_0 s}\tilde{N}(\lambda_j ds,dz_j)\Big|\Bigg]\\
&+4\sum_{j=1}^{d}E\Bigg[\sup_{0\leq t\leq T}\Big|\int_0^t\left(\Delta V^i(s^-,x)\right)'\\
&\qquad \Big(\Delta\bar{\mathcal{J}}_j(s^-,x,U^i,V^i,\bar{V}^i,\tilde{V}^i,U^{i-1},V^{i-1},\bar{V}^{i-1},\tilde{V}^{i-1})\\
&\qquad +(\Delta\bar{V}^{i-1})_j(s^-,x)-(\Delta\bar{V}^i)_j(s^-,x)\Big)e^{2\gamma_0 s}dW_j(s)\Big|\Bigg]\\
&+4\sum_{j=1}^{h}E\Bigg[\sup_{0\leq t\leq T}\Big|\int_0^t\int_{\mathcal{Z}}\left(\Delta V^i(s^-,x)\right)'\\
&\qquad \Big(\Delta\bar{\mathcal{I}}_j(s^-,x,U^i,V^i,\bar{V}^i,\tilde{V}^i,U^{i-1},V^{i-1},\bar{V}^{i-1},\tilde{V}^{i-1},z_j)\\
&\qquad +(\Delta\tilde{V}^{i-1})_j(s^-,x,z_j)-(\Delta\tilde{V}^i)_j(s^-,x,z_j)\Big)e^{2\gamma_0 s}\tilde{N}(\lambda_j ds,dz_j)\Big|\Bigg].
\end{aligned}$$

By the Burkholder-Davis-Gundy's inequality (see, e.g., Theorem 48 in page 193 of Protter [46]), the right-hand side of the inequality in (5.55) is bounded by

$$
\begin{aligned}
(5.56)\quad K_{b,0}\Bigg(&\sum_{j=1}^{d}E\Bigg[\Bigg(\int_0^T\left\|\Delta U^i(s^-,x)\right\|^2\\
&\left\|(\Delta\mathcal{J}^i)_j(s^-,x,U^i,V^i,\bar{V}^i,\tilde{V}^i,U^{i-1},V^{i-1},\bar{V}^{i-1},\tilde{V}^{i-1})\right\|^2e^{4\gamma_0 s}ds\Bigg)^{\frac{1}{2}}\Bigg]\\
&+\sum_{j=1}^{h}E\Bigg[\Bigg(\int_0^T\int_{\mathcal{Z}}\left\|\Delta U^i(s^-,x)\right\|^2\\
&\left\|\Delta\mathcal{I}_j(s^-,x,U^i,V^i,\bar{V}^i,\tilde{V}^i,U^{i-1},V^{i-1},\bar{V}^{i-1},\tilde{V}^{i-1},z_j)\right\|^2e^{4\gamma_0 s}\lambda_j\nu_j(dz_j)ds\Bigg)^{\frac{1}{2}}\Bigg]\\
&+\sum_{j=1}^{d}E\Bigg[\Bigg(\int_0^T\left\|\Delta V^i(s^-,x)\right\|^2\\
&\Big\|(\Delta\bar{\mathcal{J}}^i)_j(s^-,x,U^i,V^i,\bar{V}^i,\tilde{V}^i,U^{i-1},V^{i-1},\bar{V}^{i-1},\tilde{V}^{i-1})\\
&+(\Delta\bar{V}^{i-1})_j(s^-,x)-(\Delta\bar{V}^i)_j(s^-,x)\Big\|^2e^{4\gamma_0 s}ds\Bigg)^{\frac{1}{2}}\Bigg]\\
&+\sum_{j=1}^{h}E\Bigg[\Bigg(\int_0^T\int_{\mathcal{Z}}\left\|\Delta V^i(s^-,x)\right\|^2\\
&\Big\|\Delta\bar{\mathcal{I}}_j(s^-,x,U^i,V^i,\bar{V}^i,\tilde{V}^i,U^{i-1},V^{i-1},\bar{V}^{i-1},\tilde{V}^{i-1},z_j)\\
&+(\Delta\tilde{V}^i)_j(s^-,x,z_j)-(\Delta\tilde{V}^i)_j(s^-,x,z_j)\Big\|^2e^{4\gamma_0 s}\lambda_j\nu_j(dz_j)ds\Bigg)^{\frac{1}{2}}\Bigg]\Bigg),
\end{aligned}
$$

where $K_{b,0}$ is some nonnegative constant depending only on $K_{D,0}$ and $T$. Furthermore, it follows from the direct observation that the quantity in (5.56) is bounded by

$$
\begin{aligned}
(5.57)\qquad K_{b,0}\Bigg(&E\Bigg[\left(\sup_{0\leq t\leq T}\|\Delta U^i(t,x)\|^2e^{2\gamma_0 t}\right)^{\frac{1}{2}}\\
&\Bigg(\sum_{j=1}^{d}\left(\int_0^T\left\|\Delta\mathcal{J}_j(s^-,x,U^i,V^i,\bar{V}^i,\tilde{V}^i,U^{i-1},V^{i-1},\bar{V}^{i-1},\tilde{V}^{i-1})\right\|^2e^{2\gamma_0 s}ds\right)^{\frac{1}{2}}\\
&+\sum_{j=1}^{h}\Bigg(\int_0^T\int_{\mathcal{Z}}\left\|\Delta\mathcal{I}_j(s^-,x,U^i,V^i,\bar{V}^i,\tilde{V}^i,U^{i-1},V^{i-1},\bar{V}^{i-1},\tilde{V}^{i-1},z_j)\right\|^2\\
&e^{2\gamma_0 s}\lambda_j\nu_j(dz_j)ds\Bigg)^{\frac{1}{2}}\Bigg)\Bigg]\\
+\quad&\Bigg(E\Bigg[\left(\sup_{0\leq t\leq T}\|\Delta V^i(t,x)\|^2e^{2\gamma_0 t}\right)^{\frac{1}{2}}
\end{aligned}
$$

$$\left(\sum_{j=1}^{d}\left(\int_0^T \left\|\Delta\bar{\mathcal{J}}_j(s^-,x,U^i,V^i,\bar{V}^i,\tilde{V}^i,U^{i-1},V^{i-1},\bar{V}^{i-1},\tilde{V}^{i-1})\right.\right.\right.$$

$$\left.\left.+(\Delta\bar{V}^{i-1})_j(s^-,x)-(\Delta\bar{V}^i)_j(s^-,x)\right\|^2 e^{2\gamma_0 s}ds\right)^{\frac{1}{2}}$$

$$+\sum_{j=1}^{h}\left(\int_0^T\int_{\mathcal{Z}}\left\|\Delta\bar{\mathcal{I}}_j(s^-,x,U^i,V^i,\bar{V}^i,\tilde{V}^i,U^{i-1},V^{i-1},\bar{V}^{i-1},\tilde{V}^{i-1},z_j)\right.\right.$$

$$\left.\left.\left.\left.\left.+(\Delta\tilde{V}^{i-1})_j(s^-,x,z_j)-(\Delta\tilde{V}^i)_j(s^-,x,z_j)\right\|^2 e^{2\gamma_0 s}\lambda_j\nu_j(dz_j)ds\right)^{\frac{1}{2}}\right)\right]\right]\right).$$

In addition, by the direct computation, we know that the quantity in (5.57) is dominated by

$$\begin{aligned}
(5.58)\qquad &\frac{1}{2}E\left[\sup_{0\le t\le T}\|\Delta U^i(t,x)\|^2 e^{2\gamma_0 t}\right]\\
&+dK_{b,0}^2E\left[\int_0^T \operatorname{Tr}\left(\Delta\mathcal{J}(s^-,x,U^i,V^i,\bar{V}^i,\tilde{V}^i,U^{i-1},V^{i-1},\bar{V}^{i-1},\tilde{V}^{i-1}\right)e^{2\gamma_0 s}ds\right]\\
&+K_{b,0}^2E\left[\sum_{j=1}^{h}\int_0^T\int_{\mathcal{Z}}\operatorname{Tr}\left(\Delta\mathcal{I}(s^-,x,U^i,V^i,\bar{V}^i,\tilde{V}^i,U^{i-1},V^{i-1},\bar{V}^{i-1},\tilde{V}^{i-1},z_j)\right)_j\right.\\
&\qquad\left. e^{2\gamma_0 s}\lambda_j\nu_j(dz_j)ds\right]\\
&+\frac{1}{2}E\left[\sup_{0\le t\le T}\|\Delta V^i(t,x)\|^2 e^{2\gamma_0 t}\right]\\
&+dK_{b,0}^2E\left[\left(\int_0^T \operatorname{Tr}\left(\Delta\bar{\mathcal{J}}(s^-,x,U^i,V^i,\bar{V}^i,\tilde{V}^i,U^{i-1},V^{i-1},\bar{V}^{i-1},\tilde{V}^{i-1})\right.\right.\right.\\
&\qquad\left.\left.\left.+\Delta\bar{V}^{i-1}(s^-,x)-\Delta\bar{V}^i(s^-,x)\right)e^{2\gamma_0 s}ds\right)\right]\\
&+K_{b,0}^2E\left[\sum_{j=1}^{h}\int_0^T\int_{\mathcal{Z}}\operatorname{Tr}\left(\Delta\bar{\mathcal{I}}(s^-,x,U^i,V^i,\bar{V}^i,\tilde{V}^i,U^{i-1},V^{i-1},\bar{V}^{i-1},\tilde{V}^{i-1},z_j)\right.\right.\\
&\qquad\left.\left.+\Delta\tilde{V}^{i-1}(s^-,x,z)-\Delta\tilde{V}^i(s^-,x,z_j)\right)e^{2\gamma_0 s}\lambda_j\nu_j(dz_j)ds\right].
\end{aligned}$$

Due to (5.54), the quantity in (5.58) is bounded by

$$\begin{aligned}
(5.59)\qquad &\frac{1}{2}\left(E\left[\sup_{0\le t\le T}\|\Delta U^i(t)\|^2_{C^0(r)}e^{2\gamma_0 t}\right]+E\left[\sup_{0\le t\le T}\|\Delta U^i(t)\|^2_{C^0(q)}e^{2\gamma_0 t}\right]\right)\\
&+\hat{\gamma}_0(T+1)dK_{a,0}K_{b,0}^2\left\|(\Delta U^{i-1},\Delta V^{i-1},\Delta\bar{V}^{i-1},\Delta\tilde{V}^{i-1})\right\|^2_{\mathcal{M}^D_{\gamma_0,k}},
\end{aligned}$$

where $K_{a,0}$ is some nonnegative constant depending only on $T$, $d$, and $K_{D,0}$. Thus, it follows from (2.17) and (5.50)-(5.59) that

$$(5.60)\qquad E\left[\sup_{0\le t\le T}\left\|\Delta U^i(t)\right\|^2_{C^0(q)}e^{2\gamma_0 t}\right]+E\left[\sup_{0\le t\le T}\left\|\Delta V^i(t)\right\|^2_{C^0(q)}e^{2\gamma_0 t}\right]$$

$$\leq 2\left(1+dK_{b,0}^2\right)K_{a,0}\hat{\gamma}_0(T+1)\left\|(\Delta U^{i-1},\Delta V^{i-1},\Delta\bar{V}^{i-1},\Delta\tilde{V}^{i-1})\right\|^2_{\mathcal{M}^D_{\gamma_0,k}}.$$

Furthermore, it follows from (5.50) and (2.17) that, for $i\in\{3,4,...\}$,

$$\begin{aligned}
(5.61)\qquad & E\left[\int_t^T \mathrm{Tr}\left(\Delta\bar{V}^i(s,x)\right)e^{2\gamma_0 s}ds\right]\\
\leq\ & 2E\left[\int_t^T \mathrm{Tr}\left(\Delta\bar{\mathcal{J}}(s^-,x,U^i,V^i,\bar{V}^i,\tilde{V}^i,U^{i-1},V^{i-1},\bar{V}^{i-1},\tilde{V}^{i-1})\right.\right.\\
&\left.\left.+\Delta\bar{V}^{i-1}(s^-,x)-\Delta\bar{V}^i(s,x)\right)e^{2\gamma_0 s}ds\right]\\
&+2E\left[\int_t^T \mathrm{Tr}\left(\Delta\bar{\mathcal{J}}(s^-,x,U^i,V^i,\bar{V}^i,\tilde{V}^i,U^{i-1},V^{i-1},\bar{V}^{i-1},\tilde{V}^{i-1})\right.\right.\\
&\left.\left.+\Delta\bar{V}^{i-1}(s^-,x)\right)e^{2\gamma_0 s}ds\right]\\
\leq\ & 2\hat{\gamma}_0K_{C,0}\left(\left\|(\Delta U^{i-1},\Delta V^{i-1},\Delta\bar{V}^{i-1},\Delta\tilde{V}^{i-1})\right\|^2_{\mathcal{M}^D_{\gamma_0,k}}\right.\\
&\left.+\left\|(\Delta U^{i-2},\Delta V^{i-2},\Delta\bar{V}^{i-2},\Delta\tilde{V}^{i-2})\right\|^2_{\mathcal{M}^D_{\gamma_0,k}}\right),
\end{aligned}$$

where $K_{C,0}$ is some nonnegative constant depending only on $K_{D,0}$ and $T$. Similarly, it follows from (2.18) that

$$\begin{aligned}
(5.62)\qquad & E\left[\sum_{j=1}^h\int_t^T\int_{\mathcal{Z}}\left(\mathrm{Tr}\left(\Delta\tilde{V}^i(s^-,x,z)\right)\right)_j e^{2\gamma_0 s}\lambda_j ds\nu_j(dz_j)\right]\\
\leq\ & 2E\left[\sum_{j=1}^h\int_t^T\int_{\mathcal{Z}}\left(\mathrm{Tr}\left(\Delta\bar{\mathcal{I}}(s,x,U^i,V^i,\bar{V}^i,\tilde{V}^i,U^{i-1},V^{i-1},\bar{V}^{i-1},\tilde{V}^{i-1},z)\right.\right.\right.\\
&\left.\left.\left.+\Delta\tilde{V}^{i-1}(s^-,x,z)-\Delta\tilde{V}^i(s^-,x,z)\right)\right)_j e^{2\gamma_0 s}\lambda_j ds\nu_j(dz_j)\right]\\
&+2E\left[\sum_{j=1}^h\int_t^T\int_{\mathcal{Z}}\left(\mathrm{Tr}\left(\Delta\bar{\mathcal{I}}(s,x,U^i,V^i,\bar{V}^i,\tilde{V}^i,U^{i-1},V^{i-1},\bar{V}^{i-1},\tilde{V}^{i-1},z)\right.\right.\right.\\
&\left.\left.\left.+\Delta\tilde{V}^{i-1}(s^-,x,z)\right)\right)_j e^{2\gamma_0 s}\lambda_j ds\nu_j(dz_j)\right]\\
\leq\ & 2\hat{\gamma}_0K_{C,0}\left(\left\|(\Delta U^{i-1},\Delta V^{i-1},\Delta\bar{V}^{i-1},\Delta\tilde{V}^{i-1})\right\|^2_{\mathcal{M}^D_{\gamma_0,k}}\right.\\
&\left.+\left\|(\Delta U^{i-2},\Delta V^{i-2},\Delta\bar{V}^{i-2},\Delta\tilde{V}^{i-2})\right\|^2_{\mathcal{M}^D_{\gamma_0,k}}\right).
\end{aligned}$$

Thus, by (5.50), (5.60)-(5.62), and the fact that all functions and norms used in this paper are continuous in terms of $x$, we have

$$(5.63)\qquad \left\|(\Delta U^i,\Delta V^i,\Delta\bar{V}^i,\Delta\tilde{V}^i)\right\|^2_{\mathcal{M}^D_{\gamma_0,0}}$$

$$\leq \hat{\gamma}_0 K_{d,0}\left(\left\|(\Delta U^{i-1},\Delta V^{i-1},\Delta\bar{V}^{i-1},\Delta\tilde{V}^{i-1})\right\|^2_{\mathcal{M}^D_{\gamma_0,k}}\right.$$
$$\left.+\left\|(\Delta U^{i-2},\Delta V^{i-2},\Delta\bar{V}^{i-2},\Delta\tilde{V}^{i-2})\right\|^2_{\mathcal{M}^D_{\gamma_0,k}}\right),$$

where $K_{d,0}$ is some nonnegative constant depending only on $K_{D,0}$ and $T$.

Now, by Lemma 5.2 and the similar construction as in (5.49), for each $c \in \{1,2,...\}$, we can define

$$\zeta(\Delta U^{c,i}(t,x)+\Delta V^{c,i}(t,x)) \equiv \left(\mathrm{Tr}\left(\Delta U^{c,i}(t,x)\right)+\mathrm{Tr}\left(\Delta V^{c,i}(t,x)\right)\right)e^{2\gamma_c t}, \tag{5.64}$$

where

$$\begin{aligned}\Delta U^{c,i}(t,x)) &= (\Delta U^{(0),i}(t,x)),\Delta U^{(1),i}(t,x)),...,\Delta U^{(c),i}(t,x))',\\ \Delta V^{c,i}(t,x)) &= (\Delta V^{(0),i}(t,x)),\Delta V^{(1),i}(t,x)),...,\Delta V^{(c),i}(t,x))'.\end{aligned}$$

Then, it follows from the Itô's formula and the similar discussion for (5.63) that

$$\begin{aligned}&\left\|(\Delta U^{i},\Delta V^{i},\Delta\bar{V}^{i},\Delta\tilde{V}^{i})\right\|^2_{\mathcal{M}^D_{\gamma_c,c}} \\ \leq\ & \hat{\gamma}_c K_{d,c}\left(\left\|(\Delta U^{i-1},\Delta V^{i-1},\Delta\bar{V}^{i-1},\Delta\tilde{V}^{i-1})\right\|^2_{\mathcal{M}^D_{\gamma_c,k+c}}\right. \\ &\left.+\left\|(\Delta U^{i-2},\Delta V^{i-2},\Delta\bar{V}^{i-2},\Delta\tilde{V}^{i-2})\right\|^2_{\mathcal{M}^D_{\gamma_c,k+c}}\right) \\ \leq\ & \frac{\delta}{((c+1)^{10}(c+2)^{10}...(c+k)^{10})(\eta(c+1)\eta(c+2)...\eta(c+k))} \\ &\left(\left\|(\Delta U^{i-1},\Delta V^{i-1},\Delta\bar{V}^{i-1},\Delta\tilde{V}^{i-1})\right\|^2_{\mathcal{M}^D_{\gamma_{k+c},k+c}}\right. \\ &\left.+\left\|(\Delta U^{i-2},\Delta V^{i-2},\Delta\bar{V}^{i-2},\Delta\tilde{V}^{i-2})\right\|^2_{\mathcal{M}^D_{\gamma_{k+c},k+c}}\right),\end{aligned} \tag{5.65}$$

where, for the last inequality of (5.65), we have taken the number sequence $\gamma$ such that $\gamma_0 < \gamma_1 < ...$ and

$$\hat{\gamma}_c K_{d,c}((c+1)^{10}(c+2)^{10}...(c+k)^{10})(\eta(c+1)\eta(c+2)...\eta(c+k)) \leq \delta$$

for some $\delta > 0$ such that $2\sqrt{e^k\delta}$ is sufficiently small. Hence, we have

$$\begin{aligned}&\left\|(\Delta U^{i},\Delta V^{i},\Delta\bar{V}^{i},\Delta\tilde{V}^{i})\right\|^2_{\mathcal{M}^D_{\gamma}} \\ \leq\ & e^k\delta\left(\left\|(\Delta U^{i-1},\Delta V^{i-1},\Delta\bar{V}^{i-1},\Delta\tilde{V}^{i-1})\right\|^2_{\mathcal{M}^D_{\gamma}}\right. \\ &\left.+\left\|(\Delta U^{i-2},\Delta V^{i-2},\Delta\bar{V}^{i-2},\Delta\tilde{V}^{i-2})\right\|^2_{\mathcal{M}^D_{\gamma}}\right).\end{aligned} \tag{5.66}$$

Since $(a^2+b^2)^{1/2} \leq a+b$ for $a, b \geq 0$, we have

$$
\begin{aligned}
&\left\|(\Delta U^i, \Delta V^i, \Delta \bar{V}^i, \Delta \tilde{V}^i)\right\|_{\mathcal{M}_\gamma^D} \\
\leq\ & \sqrt{e^k \delta}\left(\left\|(\Delta U^{i-1}, \Delta V^{i-1}, \Delta \bar{V}^{i-1}, \Delta \tilde{V}^{i-1})\right\|_{\mathcal{M}_\gamma^D}\right. \\
&\left. + \left\|(\Delta U^{i-2}, \Delta V^{i-2}, \Delta \bar{V}^{i-2}, \Delta \tilde{V}^{i-2})\right\|_{\mathcal{M}_\gamma^D}\right).
\end{aligned} \tag{5.67}
$$

Therefore, by (5.67), we know that

$$
\begin{aligned}
&\sum_{i=3}^{\infty}\left\|(\Delta U^i, \Delta V^i, \Delta \bar{V}^i, \Delta \tilde{V}^i)\right\|_{\mathcal{M}_\gamma^D} \\
\leq\ & \frac{\sqrt{e^k \delta}}{1-2\sqrt{e^k \delta}}\left(2\left\|(\Delta U^2, \Delta V^2, \Delta \bar{V}^2, \Delta \tilde{V}^2)\right\|_{\mathcal{M}_\gamma^D}\right. \\
&\left. + \left\|(\Delta U^1, \Delta V^1, \Delta \bar{V}^1, \Delta \tilde{V}^1)\right\|_{\mathcal{M}_\gamma^D}\right) \\
<\ & \infty.
\end{aligned} \tag{5.68}
$$

Thus, from (5.68), we see that $(U^i, V^i, \bar{V}^i, \tilde{V}^i)$ with $i \in \{1, 2, ...\}$ forms a Cauchy sequence in $\mathcal{M}_\gamma^D[0,T]$, which implies that there is some $(U, V, \bar{V}, \tilde{V})$ such that

$$
(U^i, V^i, \bar{V}^i, \tilde{V}^i) \to (U, V, \bar{V}, \tilde{V}) \text{ as } i \to \infty \text{ in } \mathcal{M}_\gamma^D[0,T]. \tag{5.69}
$$

Finally, by (5.69) and the similar procedure as used for Theorem 5.2.1 in pages 68-71 of $\emptyset$ksendal [39], we can complete the proof of Lemma 5.3. $\square$

## 5.2 Proof of Theorem 2.1

By combining Lemmas 5.1-5.3, we can reach a proof for Theorem 2.1. $\square$

## 5.3 Proof of Theorem 2.2

First, we consider a real-valued system corresponding to the case that $\tau = T$, whose proof is along the line of the one for Lemma 5.3. More precisely, for any given number sequence $\gamma = \{\gamma_{D_c}, c = 0, 1, 2, ...\}$ with $\gamma_{D_c} \in R$, replace the norm for the Banach space $\mathcal{M}_\gamma^D[0,T]$ defined in (5.46) by the one

$$
\left\|(U, V, \bar{V}, \tilde{V})\right\|^2_{\mathcal{M}_\gamma^D} \equiv \sum_{c=0}^{\infty} \xi(c)\left\|(U, V, \bar{V}, \tilde{V})\right\|^2_{\mathcal{M}^{D_c}_{\gamma_{D_c}, c}}, \tag{5.70}
$$

for any given $(U, V, \bar{V}, \tilde{V})$ in this space, where

$$
\left\|(U, V, \bar{V}, \tilde{V})\right\|^2_{\mathcal{M}^{D_c}_{\gamma_{D_c}}} = E\left[\sup_{0 \leq t \leq T}\|U(t)\|^2_{C^c(D_c, r)} e^{2\gamma_{D_c} t}\right]
$$

$$
\begin{aligned}
&+E\left[\sup_{0\le t\le T}\|V(t)\|^2_{C^c(D_c,q)}e^{2\gamma_{D_c}t}\right]\\
&+E\left[\int_0^T\left\|\bar{V}(t)\right\|^2_{C^c(D_c,qd)}e^{2\gamma_{D_c}t}dt\right]\\
&+E\left[\int_0^T\left\|\tilde{V}(t)\right\|^2_{\nu,c}e^{2\gamma_{D_c}t}dt\right].
\end{aligned}
$$

Then, it follows from the similar argument used for (5.66) in the proof of Lemma 5.3 that

$$
(U^1(\cdot,x),V^1(\cdot,x),\bar{V}^1(\cdot,x),\tilde{V}^1(\cdot,x,z))\in\bar{\mathcal{Q}}^2_{\mathcal{F}}([0,T]\times D)
$$

with $(U^0,V^0,\bar{V}^0,\tilde{V}^0)=(0,0,0,0)$, where $(U^1,V^1,\bar{V}^1,\tilde{V}^1)$ is defined through (5.2) in Lemma 5.1. Furthermore, over each $D_c$ with $c\in\{0,1,...\}$, we have that

$$
\begin{aligned}
&\left\|(\Delta U^i,\Delta V^i,\Delta\bar{V}^i,\Delta\tilde{V}^i)\right\|^2_{\mathcal{M}^D_\gamma}\\
\le\;& e^k\delta\left(\left\|(\Delta U^{i-1},\Delta V^{i-1},\Delta\bar{V}^{i-1},\Delta\tilde{V}^{i-1})\right\|^2_{\mathcal{M}^D_\gamma}\right.\\
&\left.+\left\|(\Delta U^{i-2},\Delta V^{i-2},\Delta\bar{V}^{i-2},\Delta\tilde{V}^{i-2})\right\|^2_{\mathcal{M}^D_\gamma}\right),
\end{aligned}
\tag{5.71}
$$

where $\delta$ is a constant that can be determined by suitably choosing a number sequence $\gamma$ such that $\gamma_{D_0}<\gamma_{D_1}<...$ and $0<\sqrt{e^k\delta}/(1-2\sqrt{e^k\delta})<1$ (note that $\gamma_{D_c}$ may depend on both $D_c$ and $c$ for each $c\in\{0,1,...\}$). Thus, it follows from (5.71) that the remaining justification for Theorem 2.2 can be conducted along the line of proof for Theorem 2.1.

Second, we consider a real-valued system corresponding to the case that $\tau$ is a general stopping time. The proof for this case can be accomplished by extending the proof corresponding to $\tau=T$ via the techniques developed in Dai [16, 20] for both forward and backward SDEs, and the related discussions in Yong and Zhou [52].

Third, by direct generalizing the discussion concerning the real-valued system to complex-valued system, we reach a proof for Theorem 2.2. □

## 6 Proofs of Theorem 3.1 and Theorem 3.2

To provide the proofs for Theorem 3.1 and Theorem 4.1, we first recall the Skorohod problem and study some related properties.

### 6.1 The Skorohod Problem

Let $D([0,T],R^b)$ with $b\in\{p,2p\}$ be the space of all functions $z:[0,T]\to R^b$ that are right-continuous with left limits and are endowed with Skorohod topology (see, e.g., Billingsley [3], Jacod and Shiryaev [28]). Then, we can introduce the Skorohod problem as follows.

**Definition 6.1** *(The Skorohod problem). Given $z \in D([0,T], R^p)$ with $z(0) \in D$, a $(D,R)$-regulation of $z$ over [0,T] is a pair $(x,y) \in D([0,T], D) \times D([0,T], R^b_+)$ such that*

$$x(t) = z(t) + Ry(t) \ \ \textit{for all} \ \ t \in [0,T],$$

*where, for each $i \in \{1, ..., b\}$,*

1. *$y_i(0) = 0$,*
2. *$y_i$ is nondecreasing,*
3. *$y_i$ can increase only at a time $t \in [0,T]$ with $x(t) \in F_i$.*

Furthermore, we define the modulus of continuity with respect to a function $z(\cdot) \in D([0,T], R^b)$ and a real number $\delta > 0$ by

$$w(z, \delta, T) \equiv \inf_{t_l} \max_{l} \text{Osc}\left(z, [t_{l-1}, t_l)\right), \tag{6.1}$$

where the infimum takes over all the finite sets $\{t_l\}$ of points satisfying $0 = t_0 < t_1 < ... < t_m = T$ and $t_l - t_{l-1} > \delta$ for $l = 1, ..., m$, and

$$\text{Osc}(z, [t_{l-1}, t_l]) = \sup_{t_1 \le s \le t \le t_2} \|z(t) - z(s)\|. \tag{6.2}$$

Then, we have the following lemma.

**Lemma 6.1** *Suppose that the reflection matrix $R$ in Definition satisfies the completely-$\mathcal{S}$ condition. Then, any $(D,R)$-regulation $(x,y)$ of $z \in D([0,T], R^p)$ with $z(0) \in D$ satisfies the oscillation inequality over $[t_1, t_2]$ with $t_1, t_2 \in [0,T]$*

$$Osc(x, [t_1, t_2]) \ \le \ \kappa Osc(z, [t_1, t_2]), \tag{6.3}$$

$$Osc(y, [t_1, t_2]) \ \le \ \kappa Osc(z, [t_1, t_2]), \tag{6.4}$$

*where $\kappa$ is some nonnegative constant depending only on the inward normal vector $N$ and the reflection matrix $R$.*

Proof. For each $t \in [t_1, t_2]$, define

$$\Delta z(t) \ \equiv \ z(t) - z(t^-), \tag{6.5}$$

$$\Delta x(t) \ \equiv \ x(t) - x(t^-), \tag{6.6}$$

$$\Delta y(t) \ \equiv \ y(t) - y(t^-). \tag{6.7}$$

Since the reflection matrix $R$ satisfies the completely-$\mathcal{S}$ condition, it is easy to check that the linear complementarity problem (LCP)

$$\begin{aligned} \Delta x(t) &= \Delta z(t) + R\Delta y(t), \\ \Delta x(t) &\in D, \\ \Delta y(t) &\ge 0, \\ \Delta x_i(t)\Delta y_i(t) &= 0 \ \text{ for } \ i = 1, ..., p, \\ (b_i - \Delta x_i(t))\Delta y_i(t) &= 0 \ \text{ for } \ i = p+1, ..., b, \end{aligned}$$

is completely solvable (see also Theorem 2.1 in Mandelbaum [35] for the related discussion). Furthermore, we can conclude that

$$\Delta y(t) \leq C\Delta z(t) \tag{6.8}$$

for some nonnegative constant $C$ depending only on the inward normal vector $N$ and the reflection matrix $R$. Then, the rest of the proof is the direct conclusion of the one for Theorem 3.1 in Dai [14] or the one for Theorem 4.2 in Dai and Dai [12]. □

**Lemma 6.2** *Assume that* $(x^n, y^n) \to (x, y)$ *along* $n \in \{1, 2, ...\}$ *in* $D([0,T], R^p)\times D([0,T], R^b)$ *and* $y^n(\cdot)$ *is of bounded variation for each* $n \in \{1, 2, ...\}$*. Furthermore, suppose that*

$$\int_0^t f(x^n(s))dy^n(s) = 0 \tag{6.9}$$

*for all* $n \in \{1, 2, ...\}$ *and each* $t \in [0,T]$*, where* $f \in C^b([0,T], R^b)$ *is a b-dimensional bounded vector function. Then, for each* $t \in [0,T]$*, we have that*

$$\int_0^t f(x(s))dy(s) = 0. \tag{6.10}$$

Proof. It follows from the definition in pages 123-124 of Billingsley [3] or Theorem 1.14 in page 328 of Jacod and Shiryaev [28] that there is a sequence $\{\gamma_n, n \in \{1, 2, ...\}\}$ of continuous and strictly increasing functions mapping from $[0,T] \to [0,T]$ with $\gamma_n(0) = 0$ and $\gamma_n(T) = T$ such that

$$\sup_{t\in[0,T]} |\gamma_n(t) - t| \to 0, \tag{6.11}$$

$$\sup_{t\in[0,T]} |(x^n, y^n)(\gamma_n(t)) - (x, y)(t)| \to 0. \tag{6.12}$$

Then, by the uniform convergence in (6.11)-(6.12) and the condition in (6.9), we know that

$$\begin{aligned} \int_0^t f(x(s))dy(s) &= \lim_{n\to\infty} \int_0^t f(x^n(\gamma_n(s)))dy^n(\gamma_n(s)) \\ &= \lim_{n\to\infty} \int_0^{\gamma_n^{-1}(t)} f(x^n(u))dy^n(u) \\ &= 0, \end{aligned}$$

where $\gamma_n^{-1}(\cdot)$ is the inverse function of $\gamma_n(\cdot)$ for each $n \in \{1, 2, ...\}$. Hence, we complete the proof of Lemma 6.2. □

## 6.2 Proof of Theorem 3.1

We divide the proof of the theorem into four parts: Part A (Existence, Uniqueness), Part B, Part C, and Part D, which correspond to different boundary reflection conditions.

**Part A (Existence).** We consider the case that $L(t,\omega)$ appeared in (3.1)-(3.2) is a constant and both of the forward and the backward SDEs have reflection boundaries. In this case, we need to prove the claim that there is an adapted weak solution $((X,Y),(V,\bar{V},\tilde{V},F))$ to the system in (1.3).

In fact, for a positive integer $b$, let $D^2_{\mathcal{F}}([0,T],R^b)$ be the space of $R^b$-valued and $\{\mathcal{F}_t\}$-adapted processes with sample paths in $D([0,T],R^b)$. Furthermore, each $Y\in D^2_{\mathcal{F}}([0,T],R^b)$ is square-integrable in the sense that

$$E\left[\int_0^T\|Y(t)\|^2dt\right]<\infty. \tag{6.13}$$

In addition, we use $D^2_{\mathcal{F},p}([0,T],R^b)$ to denote the corresponding predictable space. Then, for a given $n\in\{1,2,...\}$ and a 4-tuple

$$\begin{aligned}(X^n,V^n,\bar{V}^n,\tilde{V}^n)\quad\in\quad & D^2_{\mathcal{F}}([0,T],R^p)\times D^2_{\mathcal{F}}([0,T],R^q)\times D^2_{\mathcal{F},p}([0,T],R^{q\times d})\\ &\times D^2_{\mathcal{F},p}([0,T]\times R^h_+,R^{q\times h})\end{aligned} \tag{6.14}$$

with $X^n(0)\in D$ and $V^n(T)\in\bar{D}$, we have the following observation.

By the study concerning the continuous dynamic complementarity problem (DCP) in Bernard and El Kharroubi [2] (see also the related discussions in Mandelbaum [35], Reiman and Williams [47]), Theorem 2.1 (and its proof) in the current paper, there is a 6-tuple

$$\begin{aligned}&((X^{n+1},Y^{n+1}),(V^{n+1},\bar{V}^{n+1},\tilde{V}^{n+1},F^{n+1}))\\ &\in D^2_{\mathcal{F}}([0,T],R^p)\times D^2_{\mathcal{F}}([0,T],R^b)\\ &\quad\times D^2_{\mathcal{F}}([0,T],R^q)\times D^2_{\mathcal{F},p}([0,T],R^{q\times d})\\ &\quad\times D^2_{\mathcal{F},p}([0,T]\times\mathcal{Z}^h,R^{q\times h})\times D^2_{\mathcal{F}}([0,T],R^{q\times\bar{b}})\end{aligned}$$

for each $n\in\{1,2,...\}$, satisfying the properties along each sample path:

$$X^{n+1}(t)\quad=\quad X(0)+Z^n(t)+RY^{n+1}(t)\in D, \tag{6.15}$$

with

$$\begin{aligned}Z^n(t)&=Z^n_1(t)+Z^n_2(t),\\ Z^n_1(t)&=\int_0^t b(s^-,X^n(s^-),V^n(s^-),\bar{V}^n(s^-),\tilde{V}^n(s^-,\cdot),u(s^-,X^n(s^-),\cdot)ds\\ Z^n_2(t)&=\int_0^t\sigma(s^-,X^n(s^-),V^n(s^-),\bar{V}^n(s^-),\tilde{V}^n(s^-,\cdot),u(t,X^n(s^-)),z,\cdot)dW(s)\\ &\quad+\int_0^t\int_{\mathcal{Z}^h}\eta(s^-,X^n(s^-),V^n(s^-),\bar{V}^n(s^-),\tilde{V}^n(s^-,\cdot),u(s^-,X^n(s^-)),z,\cdot)\tilde{N}(ds,dz);\end{aligned}$$

and

$$V^{n+1}(t)\quad=\quad H(X^n(T))-SF^n(T)+U^n(t)+SF^{n+1}(t)\in\bar{D}, \tag{6.16}$$

with

$$U^n(t) = U_1^n(t) - U_2^n(t) - U_3^n(t),$$

where,

$$\begin{aligned}
U_1^n(t) &= \int_t^T c(s^-, X^n(s^-), V^n(s^-), \bar{V}^n(s^-), \tilde{V}^n(s^-, \cdot), u(s^-, X^n(s^-), \cdot)ds, \\
U_2^n(t) &= \int_t^T \Big(\alpha(s^-, X^n(s^-), V^n(s^-), \bar{V}^n(s^-), \tilde{V}^n(s^-, \cdot), \\
&\qquad\qquad u(s^-, X^n(s^-)), \cdot) - \bar{V}^n(s^-)\Big)\, dW(s) \\
&\quad + \int_t^T \int_{\mathcal{Z}^h} \Big(\zeta(s^-, X^n(s^-), V^n(s^-), \bar{V}^n(s^-), \tilde{V}^n(s^-, z), \\
&\qquad\qquad u(s^-, X^n(s^-)), z, \cdot) - \tilde{V}^n(s^-, z)\Big)\, \tilde{N}(ds, dz), \\
U_3^n(t) &= \int_t^T \bar{V}^{n+1}(s^-)dW(s) + \int_t^T \int_{\mathcal{Z}^h} \tilde{V}^{n+1}(s^-, z)\tilde{N}(ds, dz).
\end{aligned}$$

Furthermore, $(X^{n+1}, Y^{n+1})$ satisfies the property (3) in Definition 4.4. In other words, $Y^{n+1}$ is a $b$-dimensional $\{\mathcal{F}_t\}$-adapted process such that the $i$th component $Y_i^{n+1}$ of $Y^{n+1}$ for each $i \in \{1, ..., b\}$ $\mathbf{P}$-a.s. has the properties that $Y_i^{n+1}(0) = 0$, $Y_i^{n+1}$ is non-decreasing, and $Y_i^{n+1}$ can increase only when $X^{n+1}$ is on the boundary face $D_i$, i.e.,

$$\int_0^t I_{D_i}(X^{n+1}(s))dY_i^{n+1}(s) = Y_i^{n+1}(t) \text{ for all } t \geq 0. \tag{6.17}$$

Similarly, $(V^{n+1}, F^{n+1})$ also satisfies the property (3) in Definition 4.4. More precisely, $F^{n+1}$ is a $q$-dimensional $\{\mathcal{F}_t\}$-adapted process such that the $i$th component $F_i^{n+1}$ of $F^{n+1}$ for each $i \in \{1, ..., \bar{b}\}$ $\mathbf{P}$-a.s. has the properties that $F_i^{n+1}(0) = 0$, $F_i^{n+1}$ is non-decreasing, and $F_i^{n+1}$ can increase only when $V^{n+1}$ is on the boundary face $\bar{D}_i$, i.e.,

$$\int_0^t I_{\bar{D}_i}(V^{n+1}(s))dF_i^{n+1}(s) = F_i^{n+1}(t) \text{ for all } t \geq 0. \tag{6.18}$$

Next, we prove that the following sequence of stochastic processes along $n \in \{1, 2, ...\}$,

$$\Xi^n = ((X^{n+1}, Y^{n+1}), (V^{n+1}, \bar{V}^{n+1}, \tilde{V}^{n+1}, F^{n+1})), \ \ (X^1, V^1, \bar{V}^1, \tilde{V}^1) = 0, \tag{6.19}$$

is relatively compact in the Skorohod topology over the space

$$\begin{aligned}
\mathcal{P}[0, T] \equiv\ & D_{\mathcal{F}}^2([0, T], R^p) \times D_{\mathcal{F}}^2([0, T], R^b) \\
& \times D_{\mathcal{F}}^2([0, T], R^q) \times D_{\mathcal{F},p}^2([0, T], R^{q \times d}) \\
& \times D_{\mathcal{F},p}^2([0, T] \times \mathcal{Z}^h, R^{q \times h}) \times D_{\mathcal{F}}^2([0, T], R^{q \times \bar{b}}).
\end{aligned} \tag{6.20}$$

Along the line of Dai [14, 18], Dai and Dai [12], and by Corollary 7.4 in page 129 of Ethier and Kurtz [23], it suffices to prove the following two conditions to be true: First, for each $\epsilon > 0$ and rational $t > 0$, there is a constant $C(\epsilon, t)$ such that

$$\liminf_{n \to \infty} P\left\{ \|\Xi^n\|^2 \leq C(\epsilon, t) \right\} \geq 1 - \epsilon; \tag{6.21}$$

Second, for each $\epsilon > 0$ and $T > 0$, there is a constant $\delta > 0$ such that

$$\limsup_{n\to\infty} P\left\{w(\Xi^n, \delta, T) \geq \epsilon\right\} \leq \epsilon. \tag{6.22}$$

To prove the two conditions stated in (6.21) and (6.22), we first define the norm along each sample path

$$\|f\|_{[a,b]} = \sup_{a\leq t\leq b} \|f(t)\|$$

for each $f \in \{X^n, Z^n, U^n, (V^n, \bar{V}^n, \tilde{V}^n)\}$ and each $a, b \in [0,T]$. Then, we introduce the space for some constant $\gamma > 0$ that will be chosen and explained in the following proof,

$$\begin{aligned}\mathcal{Q}_\gamma[0,T] \equiv\ & D^2_{\mathcal{F}}([0,T], R^p) \times D^2_{\mathcal{F}}([0,T], R^q) \times D^2_{\mathcal{F},p}([0,T], R^{q\times d}) \\ & \times D^2_{\mathcal{F},p}([0,T] \times \mathcal{Z}^h, R^{q\times h})\end{aligned} \tag{6.23}$$

endowed with the norm

$$\begin{aligned}&\left\|(X, V, \bar{V}, \tilde{V})\right\|^2_{\mathcal{Q}_\gamma[0,T]} \\ \equiv\ & E\left[\sup_{t\in[0,T]} \left(\|X(t)\|^2 + \|V(t)\|^2\right) e^{2\gamma t}\right] + E\left[\int_0^T \left\|\bar{V}(t)\right\|^2 e^{2\gamma t} dt\right] \\ & + E\left[\int_0^T \left\|\tilde{V}(t,\cdot)\right\|^2_\nu e^{2\gamma t} dt\right]\end{aligned} \tag{6.24}$$

for each $(X, V, \bar{V}, \tilde{V}) \in \mathcal{Q}_\gamma[0,T]$. Thus, by Lemma 6.1, there is a positive constant $C_1$ such that

$$\begin{aligned}&\left\|(X^{n+1}, Y^{n+1})(t)\right\| \\ \leq\ & \left\|(X^{n+1}, Y^{n+1})(0)\right\| + \kappa \mathrm{Osc}(Z^n, [0,T]) \\ \leq\ & C_1\left(\|X(0)\| + \|Z^n\|_{[0,T]}\right),\end{aligned} \tag{6.25}$$

and

$$\begin{aligned}&\left\|(V^{n+1}, \bar{V}^{n+1}, \tilde{V}^{n+1}(\cdot), F^{n+1})(t)\right\| \\ \leq\ & \left\|(V^{n+1}, \bar{V}^{n+1}, \tilde{V}^{n+1}(\cdot))(t)\right\| + \left\|F^{n+1}(t)\right\| \\ \leq\ & \left\|(V^{n+1}, \bar{V}^{n+1}, \tilde{V}^{n+1}(\cdot))(T)\right\| + \left\|F^{n+1}(0)\right\| + 2\kappa \mathrm{Osc}(U^n, [0,T]) \\ \leq\ & \bar{C}_1\left(\|(V^n(T)\| + \|F^n(T)\| + \|U^n\|_{[0,T]}\right) \\ \leq\ & \bar{C}_2\left(1 + \|X^n(T)\| + \left\|U^{n-1}\right\|_{[0,T]} + \|U^n\|_{[0,T]}\right), \\ \leq\ & C_1\left(1 + \|X(0)\| + \left\|Z^{n-1}\right\|_{[0,T]} + \left\|U^{n-1}\right\|_{[0,T]} + \|U^n\|_{[0,T]}\right),\end{aligned} \tag{6.26}$$

where, $\bar{C}_1$ and $\bar{C}_2$ are some nonnegative constants. Furthermore, we have taken $(\bar{V}^{n+1}, \tilde{V}^{n+1}(\cdot))(T) = 0$ in the third equality of (6.28) since the uniqueness for the Martingale representation is in the sense of up to sets of measure zero in $(t,\omega)$ (see, e.g., Theorem 4.3.4 in page 53 of ∅ksendal [39] and Theorem 5.3.5 in page 266 of Applebaum [1]).

Thus, for each $n \in \{1, 2, ...\}$, the given linear growth constant $L \geq 0$ in (3.1), and any constant $K > LT$, it follows from the Markov's inequality that

$$P\{\|Z_1^n\|_T \geq K\} \leq \frac{LT}{K - LT} E\left[\left\|(X^n, V^n, \bar{V}^n, \tilde{V}^n(\cdot))\right\|_{[0,T]}\right]. \tag{6.27}$$

Furthermore, by Lemma 4.2.8 in page 201 of Applebaum [1] (or related theorem in page 20 of Gihman and Skorohod [24]) and the linear growth condition, we know that

$$P\{\|Z_2^n\|_T \geq K\} \leq \frac{\bar{K}}{K^2} + \frac{L^2T}{\bar{K} - L^2T} E\left[\left\|(X^n, V^n, \bar{V}^n, \tilde{V}^n(\cdot))\right\|^2_{[0,T]}\right] \tag{6.28}$$

for all nonnegative constant $\bar{K} > L^2T$. In addition, similar to the illustration of (6.27), we have that

$$P\{\|U_1^n\|_T \geq K\} \leq \frac{1}{K - LT} E\left[\left\|(X^n, V^n, \bar{V}^n, \tilde{V}^n(\cdot))\right\|_{[0,T]}\right]. \tag{6.29}$$

Next, by the similar demonstration for (6.28) and the linear growth condition, we know that

$$P\{\|U_2^n\|_T \geq K\} \leq \frac{\bar{K}}{K^2} + \frac{L^2T}{\bar{K} - L^2T} E\left[\left\|(X^n, V^n, \bar{V}^n, \tilde{V}^n(\cdot))\right\|^2_{[0,T]}\right]. \tag{6.30}$$

Furthermore, by the proof of Theorem 2.1, we have that

$$\begin{aligned} P\{\|U_3^n\|_T \geq K\} \leq\ & \frac{\bar{K}}{K^2} + \frac{\bar{K}_1 T}{(\bar{K} - L^2T)^2} \\ & + \frac{\bar{K}_2}{(\bar{K} - L^2T)^2} E\left[\left\|(X^n, V^n, \bar{V}^n, \tilde{V}^n)\right\|^2_{\mathcal{Q}_\gamma[0,T]}\right] \end{aligned} \tag{6.31}$$

for some nonnegative constants $\bar{K}_1$ and $\bar{K}_2$. Therefore, for each given $\epsilon > 0$, it follows from (6.27)-(6.31), suitably chosen constants $K$ and $\bar{K}$, and the initial condition in (6.19) that there is a nonnegative constant $C$ such that

$$\begin{aligned} & \inf_n P\{\|\Xi^n(t)\| \leq C,\ 0 \leq t \leq T\} \\ \geq\ & \inf_n \min\left\{P\left\{\left\|(X^{n+1}, Y^{n+1})(t)\right\| \leq C,\ 0 \leq t \leq T\right\},\right. \\ & \qquad \left. P\left\{\left\|(V^{n+1}, \bar{V}^{n+1}, \tilde{V}^{n+1}, F^{n+1})(t)\right\| \leq C,\ 0 \leq t \leq T\right\}\right\} \\ \geq\ & 1 - \epsilon. \end{aligned} \tag{6.32}$$

Thus, the condition in (6.21) is satisfied by the sequence of $\{\Xi^n\}$.

Now, for any $t \in [0, T]$, it follows from the proof of Proposition 18 for a BSDE with jumps in Dai [16] and Lemma 6.1 that

$$
\begin{aligned}
(6.33) \qquad & \left\|(U^n, \bar{V}^n, \tilde{V}^n)\right\|^2_{\mathcal{Q}_\gamma[t,T]} \\
\leq \;& K_\gamma \left( 2L^2(T-t) + \left\|(X^{n-1}, V^{n-1}, \bar{V}^{n-1}, \tilde{V}^{n-1})\right\|^2_{\mathcal{Q}_\gamma[t,T]} \right) \\
\leq \;& K_\gamma \left( 2L^2(T-t) + e^{2\gamma T} E\left[\|V^{n-1}\|^2_{[t,T]}\right] + e^{2\gamma T} \int_t^T E\left[\|X^{n-1}\|^2_{[0,s]}\right] ds \right) \\
& + K_\gamma \left\|(U^{n-1}, \bar{V}^{n-1}, \tilde{V}^{n-1})\right\|^2_{\mathcal{Q}_\gamma[t,T]},
\end{aligned}
$$

where, $K_\gamma < 1$ depending only on $L$, $T$, $d$, and $h$ for some suitable chosen $\gamma > 0$. Thus, by Lemma 6.1, the Itô's isometry formula, and (6.33), we have that

$$
\begin{aligned}
(6.34) \qquad & E\left[\|V^n\|^2_{[t,T]}\right] \\
\leq \;& \bar{K}_1 \left( E\left[\|V^n(T)\|^2\right] + E\left[\|F^{n-1}(T)\|^2\right] + \kappa^2 E\left[\mathrm{Osc}(U^{n-1}, [t,T])^2\right] \right) \\
\leq \;& K_1 \left( 1 + E\left[\|X^n\|^2_{[0,T]}\right] + \kappa^2 E\left[\mathrm{Osc}(U^{n-2}, [0,T])^2\right] + \kappa^2 E\left[\mathrm{Osc}(U^{n-1}, [t,T])^2\right] \right) \\
\leq \;& K_1 \left( 1 + 24\kappa^2 L^2 T^2 + 24\kappa^2 L^2 (T-t)^2 \right) + K_1 E\left[\|X^n\|^2_{[0,T]}\right] \\
& + 24 K_1 \kappa^2 L^2 T \left( \int_0^T E\left[\|X^{n-2}\|^2_{[0,s]}\right] ds + E\left[\|V^{n-2}\|^2_{[0,T]}\right] \right) \\
& + 24 K_1 \kappa^2 L^2 (T-t) \left( \int_t^T E\left[\|X^{n-1}\|^2_{[0,s]}\right] ds + E\left[\|V^{n-1}\|^2_{[t,T]}\right] \right) \\
& + 24 K_1 \kappa^2 L^2 T \left\|(U^{n-2}, \bar{V}^{n-2}, \tilde{V}^{n-2})\right\|^2_{\mathcal{Q}_\gamma[0,T]} \\
& + 4 K_1 \kappa^2 \left\|(U^{n-1}, \bar{V}^{n-1}, \tilde{V}^{n-1})\right\|^2_{\mathcal{Q}_\gamma[0,T]} \\
& + 24 K_1 \kappa^2 L^2 (T-t) \left\|(U^{n-1}, \bar{V}^{n-1}, \tilde{V}^{n-1})\right\|^2_{\mathcal{Q}_\gamma[t,T]} \\
& + 4 K_1 \kappa^2 \left\|(U^n, \bar{V}^n, \tilde{V}^n)\right\|^2_{\mathcal{Q}_\gamma[t,T]} \\
\leq \;& K_3 + K_2 \left( \int_0^T E\left[\|X^{n-1}\|^2_{[0,s]}\right] ds + \left\|(U^{n-2}, \bar{V}^{n-2}, \tilde{V}^{n-2})\right\|^2_{\mathcal{Q}_\gamma[0,T]} \right. \\
& \left. + \left\|(U^{n-1}, \bar{V}^{n-1}, \tilde{V}^{n-1})\right\|^2_{\mathcal{Q}_\gamma[0,T]} + \left\|(U^n, \bar{V}^n, \tilde{V}^n)\right\|^2_{\mathcal{Q}_\gamma[0,T]} \right),
\end{aligned}
$$

where, $K_i$ for $i \in \{1, 2, 3\}$ are some nonnegative constants depending only on $T$, $L$, $\kappa$, and $E\left[\|V(T)\|^2\right]$. Furthermore, for any $t \in [0, T]$, we have that

$$
\begin{aligned}
(6.35) \qquad E\left[\|X^n\|^2_{[0,t]}\right] &\leq 2E\left[\|X(0)\|^2\right] + 2\kappa^2 E\left[\mathrm{Osc}(Z^{n-1}, [0,t])^2\right] \\
&\leq 2E\left[\|X(0)\|^2\right] + 6\kappa^2 L^2 t^2
\end{aligned}
$$

$$
\begin{aligned}
& +6\kappa^2 L^2 t \left( \int_0^t E\left[ \left\| X^{n-1} \right\|_{[0,s]}^2 \right] ds + E\left[ \left\| V^{n-1} \right\|_{[0,T]}^2 \right] \right) \\
& +6\kappa^2 L^2 t \left\| (U^{n-1}, \bar{V}^{n-1}, \tilde{V}^{n-1}) \right\|_{\mathcal{Q}_\gamma[0,T]}^2 \\
\leq \;& 2E\left[ \| X(0) \|^2 \right] + 12\kappa^4 L^2 t^2 + 6\kappa^2 L^2 t E\left[ \| V^2(T) \| \right] \\
& +6\kappa^2 L^2 t \int_0^t E\left[ \left\| X^{n-1} \right\|_{[0,s]}^2 \right] ds \\
& +6\kappa^2 L^2 t \left( 1 + 2\kappa^2 \right) \left\| (U^{n-1}, \bar{V}^{n-1}, \tilde{V}^{n-1}) \right\|_{\mathcal{Q}_\gamma[0,T]}^2 .
\end{aligned}
$$

Therefore, for any $\epsilon > 0$ and a constant $\delta > 0$, consider a finite set $\{t_l\}$ of points satisfying $0 = t_0 < t_1 < ... < t_m = T$ and $t_l - t_{l-1} = \delta < \epsilon/L$ with $l \in \{1, ...m\}$. It follows from (6.19), (6.33)-(6.35), and the similar explanation for (6.27) that

$$
\begin{aligned}
& P\left\{ w(Z_1^n, \delta, T) \geq \epsilon \right\} \\
\leq \;& \frac{3L^2\delta}{(\epsilon - L\delta)^2} \left( E\left[ \| X^n \|_{[0,T]}^2 + \| V^n \|_{[0,T]}^2 \right] + \left\| (U^n, \bar{V}^n, \tilde{V}^n) \right\|_{\mathcal{Q}_\gamma[0,T]}^2 \right) \\
\leq \;& \frac{3L^2\delta}{(\epsilon - L\delta)^2} \left( A_0 + \sum_{k=1}^n \frac{A_1^{k+1} T^{k+1}}{(k+1)!} \left( 1 + K_\gamma^k \right) + A_2 \sum_{k=1}^n K_\gamma^k \right),
\end{aligned}
\tag{6.36}
$$

where $A_0$, $A_1$, and $A_2$ are some constants depending only on $L$, $T$, $d$, and $h$. Furthermore, by Lemma 4.2.8 in page 201 of Applebaum [1] (or related theorem in page 20 of Gihman and Skorohod [24]) and the linear growth condition, we know that

$$
\begin{aligned}
& P\left\{ w(Z_2^n, \delta, T) \geq \epsilon \right\} \\
\leq \;& \frac{\bar{\epsilon}}{\epsilon^2} + \frac{3L^2}{\bar{\epsilon} - 3L^2\delta} \left( \delta E\left[ \| X^n \|_T^2 \right] + \delta E\left[ \| V^n \|_T^2 \right] + E\left[ \left\| (U^n, \bar{V}^n, \tilde{V}^n) \right\|_{\mathcal{Q}_\gamma[0,T]}^2 \right] \right) \\
\leq \;& \frac{\bar{\epsilon}}{\epsilon^2} + \frac{3L^2}{\bar{\epsilon} - 3L^2\delta} \left( \delta \left( A_0 + \sum_{k=1}^n \frac{A_1^{k+1} T^{k+1}}{(k+1)!} \left( 1 + K_\gamma^k \right) + A_2 \sum_{k=1}^n K_\gamma^k \right) + A_3 \sum_{k=1}^n K_\gamma^k \right)
\end{aligned}
\tag{6.37}
$$

for all nonnegative constant $\bar{\epsilon} > 3L^2\delta$, where $A_3$ is some constant depending only on $L$, $T$, $d$, and $h$.

Similarly, there are some constants $B_0$, $B_1$, $B_2$, and $B_3$ depending only on $L$, $T$, $d$, and $h$ such that

$$
\begin{aligned}
& P\left\{ w(U_1^n, \delta, T) \geq \epsilon \right\} \\
\leq \;& \frac{3L^2\delta}{(\epsilon - L\delta)^2} \left( B_0 + \sum_{k=1}^n \frac{B_1^{k+1} T^{k+1}}{(k+1)!} \left( 1 + K_\gamma^k \right) + B_2 \sum_{k=1}^n K_\gamma^k \right),
\end{aligned}
\tag{6.38}
$$

and

$$
\begin{aligned}
& P\left\{ w(Z_2^n, \delta, T) \geq \epsilon \right\} \\
\leq \;& \frac{\bar{\epsilon}}{\epsilon^2} + \frac{3L^2}{\bar{\epsilon} - 3L^2\delta} \left( \delta \left( B_0 + \sum_{k=1}^n \frac{B_1^{k+1} T^{k+1}}{(k+1)!} \left( 1 + K_\gamma^k \right) + B_2 \sum_{k=1}^n K_\gamma^k \right) + B_3 \sum_{k=1}^n K_\gamma^k \right).
\end{aligned}
\tag{6.39}
$$

Hence, for each given $\epsilon > 0$, it follows from (6.36)-(6.39) and suitably chosen constants $\bar{\epsilon}$, $\delta$, and $\gamma$ that

$$\limsup_{n\to\infty} P\left\{w(\Xi^n), \delta, T) \geq \epsilon\right\} \leq \epsilon. \tag{6.40}$$

Thus, the condition in (6.22) is true for the sequence of $\{\Xi^n\}$. Hence, by (6.28), (6.40), and Corollary 7.4 in page 129 of Ethier and Kurtz [23], this sequence is relatively compact. Therefore, there is a subsequence of $\{\Xi^n\}$ that converges weakly to $\Xi \equiv ((X,Z,Y),(V,\bar{V},\tilde{V},F))$ over the space $\mathcal{P}[0,T]$. For convenience, we suppose that the subsequence is the sequence itself, i.e.,

$$\Xi^n \Rightarrow \Xi. \tag{6.41}$$

Then, by the Skorohod representation theorem (see, e.g., Theorem 1.8 in page 102 of Ethier and Kurtz [23]), we can assume that the convergence in (6.41) is a.s. in the Skorohod topology. Thus, by the claim (a) in Theorem 1.14 (or the claim (a) in Proposition 2.1) of Jacod and Shiryaev [28] and the facts that $Y^{n+1}(0) = 0$ and $Y^{n+1}$ is nondecreasing, we can conclude that $Y(0) = 0$ and $Y$ is nondecreasing. Furthermore, by Lemma 6.2 and (6.17)

$$\int_0^t I_{D_i}(X(s))dY_i(s) = Y_i(t) \text{ for all } t \geq 0,\ i \in \{1,...,b\}. \tag{6.42}$$

Similarly, we know that $F(0) = 0$, $F$ is non-decreasing, and

$$\int_0^t I_{\bar{D}_i}(V(s))dF_i(s) = F_i(t) \text{ for all } t \geq 0,\ i \in \{1,...,\bar{b}\}. \tag{6.43}$$

Therefore, by the Lipschitz condition in (3.2), we know that $((X,Y),(V,\bar{V},\tilde{V},F))$ satisfies the FB-SDEs in (1.3) a.s. Thus, by the Skorohod representation theorem again, it is a weak solution to the FB-SDEs in (1.3).

**Part A (Uniqueness).** Assume that $((X,Y),(V,\bar{V},\tilde{V},F))$ is a weak solution to the FB-SDEs in (1.3). To prove its uniqueness, we introduce some additional notations. Let $D_\emptyset = D$, $\bar{D}_\emptyset = \bar{D}$, and define

$$D_K \equiv \cap_{i\in K} D_i,\ \ \bar{D}_{\bar{K}} \equiv \cap_{i\in\bar{K}} \bar{D}_i \tag{6.44}$$

for each $\emptyset \neq K \subset \{1,...,b\}$ and each $\emptyset \neq \bar{K} \subset \{1,...,\bar{b}\}$. In the sequel, we call a set $K \in \{1,...,b\}$ "maximal" if $K \neq \emptyset$, $D_K \neq \emptyset$, and $D_K \neq D_{\tilde{K}}$ for any $\tilde{K} \supset K$ such that $\tilde{K} \neq K$. Similarly, we can define the maximal set corresponding to a set $\bar{K} \in \{1,...,\bar{b}\}$. Furthermore, let $d(x, D_K)$ and $d(\bar{x}, \bar{D}_{\bar{K}})$ respectively denote the Euclidean distance between $x$ and $D_K$ for a point $x \in D$ and the Euclidean distance between a point $\bar{x} \in \bar{D}$ and $\bar{D}_{\bar{K}}$. Then, it follows from Lemma 3.2 in Dai [14] or Lemma B.1 in Dai and Williams [13] that there exist two constants $C \geq 1$ and $\bar{C} \geq 1$ such that

$$d(x, D_K) \leq C\sum_{i\in K}(n_i \cdot x - b_i),\ \ \bar{d}(\bar{x}, \bar{D}_{\bar{K}}) \leq \bar{C}\sum_{i\in\bar{K}}(\bar{n}_i \cdot \bar{x} - \bar{b}_i). \tag{6.45}$$

Now, for each $\epsilon \geq 0$, $K \in \{1, ..., b\}$, and $\bar{K} \in \{1, ..., \bar{b}\}$ (including the empty set), we let

$$
\begin{aligned}
(6.46) \qquad D_K^\epsilon &\equiv \{x \in R^q : 0 \leq n_i \cdot x - b_i \leq C_\epsilon \text{ for all } i \in K, \\
&\qquad n_i \cdot x - b_i > \epsilon \text{ for all } i \in \{1, ..., b\} \setminus K\}, \\
(6.47) \qquad \bar{D}_{\bar{K}}^\epsilon &\equiv \{\bar{x} \in R^q : 0 \leq \bar{n}_i \cdot \bar{x} - \bar{b}_i \leq \bar{C}_\epsilon \text{ for all } i \in \bar{K}, \\
&\qquad \bar{n}_i \cdot \bar{x} - \bar{b}_i > \epsilon \text{ for all } i \in \{1, ..., \bar{b}\} \setminus \bar{K}\},
\end{aligned}
$$

where $C_\epsilon = Cp\epsilon$ and $\bar{C}_\epsilon = \bar{C}q\epsilon$. Thus, by Lemmas 4.1-4.2 in Dai and Williams [13], we know that

$$
(6.48) \qquad D = \cup_{K \in \mathcal{G}} D_K^\epsilon, \quad \bar{D} = \cup_{\bar{K} \in \bar{\mathcal{G}}} \bar{D}_{\bar{K}}^\epsilon,
$$

where, $\mathcal{G}$ is the collection of subsets of $\{1, ..., b\}$ consisting of all maximal sets in $\{1, ..., b\}$ and $\bar{\mathcal{G}}$ is defined in the same way in terms of subsets of $\{1, ..., \bar{b}\}$. For convenience, we order the sets in $\mathcal{G}$ and $\bar{\mathcal{G}}$. Then, we can define a sequence of 3-dimensional points $\{(r_n, \bar{r}_n, \tau_n), n \in \{1, 2, ...\}\}$ with $\tau_0 = 0$ by induction.

In fact, since $((X, Y), (V, \bar{V}, \tilde{V}, F))$ is a weak solution to the FB-SDEs in (1.3), both $X(0)$ and $V(0)$ are defined. Thus, if $(r_1, \bar{r}_1)$ is the first $K \times \bar{K} \in \{1, ..., b\} \times \{1, ..., \bar{b}\}$ such that $(x, \bar{x}) \in D_{r_1}^\epsilon \times \bar{D}_{\bar{r}_1}^\epsilon$, we let

$$
(6.49) \qquad \tau_1 = \inf \left\{t \geq 0 : \ (X(t), V(t)) \notin D_{r_1}^\epsilon \times \bar{D}_{\bar{r}_1}^\epsilon\right\}.
$$

Furthermore, if $(r_n, \bar{r}_n, \tau_n)$ has been defined on $\{\tau_n < \infty\}$, we let $(r_{n+1}, \bar{r}_{n+1})$ be the first $K \times \bar{K} \in \mathcal{G} \times \bar{\mathcal{G}}$ such that $(X(\tau_n), V(\tau_n)) \in D_K^\epsilon \times \bar{D}_{\bar{K}}^\epsilon$. Then, we can define

$$
(6.50) \qquad \tau_{n+1} = \inf \left\{t \geq \tau_n : \ (X(t), V(t)) \notin D_{r_{n+1}}^\epsilon \times \bar{D}_{\bar{r}_{n+1}}^\epsilon\right\}.
$$

On $\{\tau_n = +\infty\}$, we define $r_{n+1} = r_n$, $\bar{r}_{n+1} = \bar{r}_n$, and $\tau_{n+1} = \tau_n$. Due to the right-continuity of the sample paths of solution $(X, V)$ by the related property of Lévy process driven stochastic integral (see, e.g., Theorem 4.2.12 in page 204 of Applebaum [1]), $\{\tau_n\}$ is a nondecreasing sequence of $\{\mathcal{F}_t\}$-stopping times, satisfying $\tau_n \to \infty$ a.s. as $n \to \infty$.

Hence, it suffices to prove the weak uniqueness of $((X, Y), (V, \bar{V}, \tilde{V}, F))(\cdot \wedge \tau_n)$ for each $n$. Note that both $D_{r_n}^\epsilon$ and $\bar{D}_{\bar{r}_n}^\epsilon$ for each $n$ are subsets of cones. Thus, without loss of generality, we assume that both $D$ and $\bar{D}$ are cones. Therefore, we can prove the weak uniqueness by induction in terms of the numbers of boundary faces of $D$ and $\bar{D}$.

In fact, for the case that $b = \bar{b} = 1$, it follows from the uniqueness of the Skorohod mapping given by Lemma 3.1 in Dai [14] or Lemma 4.5 in Dai and Dai [12] that the weak uniqueness is true. Now, we suppose that the weak uniqueness is true for the case that $b + \bar{b} = m \geq 2$ with $b \geq 1$ and $\bar{b} \geq 1$. Then, we can prove the case for $b + \bar{b} = m + 1$. In this case, we need to consider two folds indexed by two pairs of $(b + 1, \bar{b})$ and $(b, \bar{b} + 1)$. Both of the folds can be proved by the similar discussion for Theorem 5.4 in Dai and Williams [13]. Therefore, we finish the proof of weak uniqueness.

**Part B.** We consider the case that $L(t, \omega)$ appeared in (3.1)-(3.2) is a constant and the spectral radii of $S$ and each $p \times p$ sub-principal matrix of $N'R$ are strictly less than one. In this case, we need to prove that there is a unique strong adapted solution $((X, Y), (V, \bar{V}, \tilde{V}, F))$ to the system of in (1.3).

In fact, it follows from the discussions in Reiman and Harrison [27], Dai [18], Lemma 7.1 and Theorem 7.2 in pages 164-165 of Chen and Yao [11] that there exist two Lipschitz continuous mappings $\Phi$ and $\Psi$ such that

$$(X^{n+1}, Y^{n+1}) = \Phi(Z^n) \tag{6.51}$$

$$(V^{n+1}, F^{n+1}) = \Psi(U^n) \tag{6.52}$$

for each $n \in \{1, 2, ...\}$. Then, it follows from (6.51)-(6.52), the related estimates in Part A, and the conventional Picard's iterative method, we can reach a proof for the claim in Part B.

**Part C.** We consider the case that $L(t, \omega)$ appeared in (3.1)-(3.2) is a constant and both of the SDEs have no reflection boundaries. In this case, we need to prove that there is a unique strong adapted solution $((X, Y), (V, \bar{V}, \tilde{V}, F))$ to the system of in (1.3). In fact, by the related estimates in Part A, this case can be proved by directly generalizing the conventional Picard's iterative method. Actually, this case is a special one of Theorem 2.1 or Theorem 2.2.

**Part D.** We consider the case that $L(t, \omega)$ appeared in (3.1)-(3.2) is a general adapted and mean-squarely integrable stochastic process. The proofs corresponding to the cases stated in Part A, Part B, and Part C can be accomplished along the lines of proofs for Lemma 4.1 in Dai [16] associated with a forward SDE under random environment and Proposition 18 in Dai [20] for a backward SDE under random environment. The key in the proofs is to introduce the following sequence of $\{\mathcal{F}_t\}$-stopping times, i.e.,

$$\tau_n \equiv \inf\{t > 0, \|L(t)\| > n\} \text{ for each } n \in \{1, 2, ...\}. \tag{6.53}$$

By the condition in (3.6), $\tau_n$ is nondecreasing and a.s. tends to infinity as $n \to \infty$.

Finally, by summarizing the cases presented in Part A to Part D, we finish the proof of Theorem 3.1. □

### 6.3 Proof of Theorem 3.2

For a control process $u^* \in \mathcal{C}$, it follows from Theorem 2.1 that the $(r, q+1)$-dimensional FB-SPDEs in (1.1) with the partial differential operators $\{\bar{\mathcal{L}}, \bar{\mathcal{J}}, \bar{\mathcal{I}}\}$ given by (3.7)-(3.12) and terminal condition in (3.13) indeed admits a well-posed 4-tuple solution $(U(t, x)$, $V(t, x)$, $\bar{V}(t, x)$, $\tilde{V}(t, x, \cdot))$. Thus, substituting

$$(V(t), \bar{V}(t), \tilde{V}(t, \cdot)) \equiv (V(t, X(t)), \bar{V}(t, X(t)), \tilde{V}(t, X(t), \cdot))$$

into the system of coupled FB-SDEs in (1.3), it follows from Theorem 3.1 that the claims in Theorem 3.2 are true. □

## 7 Proof of Theorem 4.1

The proof of part 1 is the direct extension of the single-dimensional case (i.e., $p = q = 1$) for the related optimal control problem in Øksendal *et al.* [41].

The proof of part 2 can be done as follows. For each $u \in \mathcal{C}$ and $\gamma(t,x) = \beta(t,x) \equiv 0$, it follows from Theorem 3.2 that the regulator processes $F(t)$ and $Y(t)$ exist. Since they are nondecreasing with respect to time variable $t$, the derivatives $\frac{dF}{dt}(t,x)$ and $\frac{dY}{dt}(t,x)$ exist a.e. in terms of time variable $t$ along each sample path a.s. Furthermore, if each $q \times q$ sub-principal matrix of $\bar{N}'S$ and each $p \times p$ sub-principal matrix of $N'R$ are invertible, these derivatives are uniquely determined owing to the Skorohod mapping. Nevertheless, if only the general completely-$\mathcal{S}$ condition is imposed, these derivatives are weakly unique in a probability distribution sense. In addition, it follows from Proposition 7.1 in Ethier and Kurtz [23] that these derivatives can be approximated by polynomials in terms of variable $x$ for each given $t$, which are denoted by $\gamma(t,x)$ and $\beta(t,x)$. Then, the proof for the claim in part 2 follows from the one for the claim in part 1. Hence, we reach a proof for Theorem 4.1. $\square$

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